Focalizing Linear Logic in Itself

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Abstract

We sketch a simple, syntactic, and modular proof of the correctness of focusing for linear logic via an interpretation back into linear logic itself. The proof proceeds first by showing proof-theoretic versions of some elementary facts that hold of any pair of adjoint functors, and then specializing to an adjunction that embodies focusing discipline.

1 Introduction

Focusing [And92] is a phenomenon discovered by Andreoli in the setting of reducing nondeterminism in proof search, but it is increasingly clear that it a deep property of all well-behaved logics.

We first observe that all the connectives of intuitionistic linear logic are either *invertible* on the left of the turnstile (like $\otimes, \oplus, 1, 0$) which is to say their left sequent rule's conclusion implies its premise(s), or else similarly invertible on the right (like $\&, \multimap, \top$).

To have a focusing discipline is to impose two sorts of requirements on proofs. An *inverting proof* is one in which, when read bottom-up, invertible rules *must* be applied eagerly, preferring right decompositions to left, and performing left decompositions in order from right to left.

A *focusing* proof, Andreoli's key novelty, goes farther still and demands not only that invertible rules (also called 'asynchronous') are applied eagerly, but also that the remaining rules — the synchronous rules — are used in uninterrupted sequences working on a single proposition until they reach asynchronous decompositions again.

The soundness of both of these restrictions is easy to see, since they amount to restrictions of ordinary proof, and completeness of inverting proofs is also quite reasonable, since we need only do a little bit of reasoning (once we have shown the cut elimination and identity theorems in the ambient logic) for each connective to see that its left or right rule can be inverted.

The completeness of focusing is traditionally more difficult, depending on unpleasant per-connective lemmas about how to permute their rules around inside larger derivations. Our aim is to deliver a simple inductive proof that only requires a 'constant' amount of effort per connective, and also brings to light what it is about a given connective that allows it to fit well into the larger focusing picture: in the end the answer is that positive connectives are the things that commute appropriately with a left adjoint F, and negative connectives are the things that commute with a right adjoint U. We begin by explaining the required notion of adjunction, prove a few results about it, and then specialize one result to the case of the 'token-passing adjunction' where $F = q^+ \otimes —$ and $U = q^+ \multimap —$ to see how the completeness (and indeed also, in passing the soundness) of focusing falls out as a corollary.

Almost none of the individual ideas in the proof are completely new. Indeed adjunctions have been a workhorse concept in category theory for decades, and Paul Levy for example has already emphasized [Lev99] the importance of thinking about specifically an adjunction between positive/value-producing types and negative/computation types.

The token-passing adjunction also appears in Lamarche's account [Lam95] of games semantics for linear logic. And Olivier Laurent's unpublished proof [Lau04] of focalization 'by cut elimination' strongly resembles a version of this proof but specialized to the classical version of linear logic, with all the reasoning about adjunctions appearing only tacitly.

However, to the best of this author's knowledge, no one has pointed out that focusing discipline is *merely* the weaker notion of inverting discipline for an interpretation of the polarity shifting operations as the token-passing adjunction. Thus while focalization is a deep and important fact, it admits a very shallow proof, one that can be carried out 'in the logic' itself by simply constructing small derivations and cutting them in at appropriate points.

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2 Adjunctions

Suppose F, U are assumed to be unary operations that compute a proposition from another proposition in linear logic, such that the following rules are admissible:

$$\frac{}{FUB \vdash B}\,\epsilon \qquad \frac{}{A \vdash UFA}\,\eta \qquad \frac{\Delta, B_1 \vdash B_2}{\Delta, UB_1 \vdash UB_2}\,U \qquad \frac{\Delta, A_1 \vdash A_2}{\Delta, FA_1 \vdash FA_2}\,F$$

In this event we say we have an *adjunction* $F \dashv U$, and F is left adjoint to U, and U is right adjoint to F. Analogues of all the usual category-theoretic results are easily obtained.

Lemma 2.1 (Adjoint Transpose) *The rule*

$$\frac{\Delta, FA \vdash B}{\Delta, A \vdash UB}\,\dashv$$

is admissible, both top-down and bottom-up.

Proof By construction of derivation and cut elimination.

$$\dfrac{\dfrac{}{\Delta, A \vdash UFA}\,\eta \qquad \dfrac{\Delta, FA \vdash B}{\Delta, UFA \vdash UB}\,U}{\Delta, A \vdash UB}\,cut \qquad \dfrac{\dfrac{\Delta, A \vdash UB}{\Delta, FA \vdash FUB}\,F \qquad \dfrac{}{\Delta, FUB \vdash B}\,\epsilon}{\Delta, FA \vdash B}\,cut$$

■

Corollary 2.2 (Triangle Equalities) *$UB \dashv\vdash UFUB$ and $FA \dashv\vdash FUFA$.*

Proof Transpose $UFA \vdash UFA$ and $FUB \vdash FUB$, and apply F to η and U to ϵ. ■

Lemma 2.3 (Adjunctions Preserve Limits) *All of the following hold:*

- $F(A_1 \otimes A_2) \dashv\vdash FA_1 \otimes A_2 \dashv\vdash A_1 \otimes FA_2$
- $U(A \multimap B) \dashv\vdash FA \multimap B \dashv\vdash A \multimap UB$
- $F(A_1 \oplus A_2) \dashv\vdash FA_1 \oplus FA_2$
- $U(B_1 \,\&\, B_2) \dashv\vdash UB_1 \,\&\, UB_2$
- $F0 \dashv\vdash 0$
- $U\top \dashv\vdash \top$

Proof By construction of derivation. We show the case of $U(A \multimap B) \dashv\vdash FA \multimap B$, as the other cases are reasonably similar.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{FA \vdash FA} \quad \dfrac{}{B \vdash B}}{FA \multimap B, FA \vdash B}}{FA \multimap B, A \vdash UB}\,\dashv}{F(FA \multimap B), A \vdash B}\,\dashv}{F(FA \multimap B) \vdash A \multimap B}}{FA \multimap B \vdash U(A \multimap B)}\,\dashv \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A} \quad \dfrac{}{B \vdash B}}{A \multimap B, A \vdash B}}{U(A \multimap B), A \vdash UB}\,U}{U(A \multimap B), FA \vdash B}\,\dashv}{U(A \multimap B) \vdash FA \multimap B}$$

■

3 Polarized Language

Consider the language of polarized linear logic propositions, including polarized atoms:

$$\begin{array}{rcll} \text{Positive Propositions} & P & ::= & {\downarrow}N \mid P \otimes P \mid P \oplus P \mid 1 \mid 0 \mid p^+ \\ \text{Negative Propositions} & N & ::= & {\uparrow}P \mid P \multimap N \mid N \,\&\, N \mid \top \mid p^- \end{array}$$

We continue to use A, B, C to denote a typical proposition that may be either positive or negative, and Γ, Δ for contexts of propositions, again allowed to be of either polarity. To provide a notion of proof for this language we may consider translations of it into one that already has a proof theory.

3.1 Translations

We define three translations into ordinary linear logic. Let A^- (the 'depolarization' of A) be the unpolarized linear logic proposition that is A with all $\uparrow$ and $\downarrow$ erased. Let $A^\bullet$ (the 'focalization' of A) be what is obtained by replacing every $\uparrow$ in A with F and every $\downarrow$ with U. Finally, define A° (the 'defocalization' of A) as follows:

$$
\begin{aligned}
&({\downarrow}N)^\circ = UN^\circ \\
&(P_1 \otimes P_2)^\circ = UFP_1^\circ \otimes UFP_2^\circ \\
&(P_1 \oplus P_2)^\circ = UFP_1^\circ \oplus UFP_2^\circ \\
&(1)^\circ = 1 \\
&(0)^\circ = 0 \\
&(p^+)^\circ = p^+ \\
&({\uparrow}P)^\circ = FP^\circ \\
&(P \multimap N)^\circ = UFP^\circ \multimap FUN^\circ \\
&(N_1 \mathbin{\&} N_2)^\circ = FUN_1^\circ \mathbin{\&} FUN_2^\circ \\
&(\top)^\circ = \top \\
&(p^-)^\circ = p^-
\end{aligned}
$$

Although this translation introduces many more instances of F and U than $—^\bullet$, it does not essentially differ in proof strength from it, this fact being captured by the following lemma.

Lemma 3.1

$$UN^\circ \dashv\vdash UN^\bullet$$

$$FP^\circ \dashv\vdash FP^\bullet$$

Proof By induction and appeal to 'adjoints preserve limits' and the triangle equalities. For example,

$$F(UFP_1^\circ \otimes UFP_2^\circ) \dashv\vdash F(UFP_1 \otimes UFP_2)$$

$$\dashv\vdash FUFP_1 \otimes UFP_2 \dashv\vdash FP_1 \otimes UFP_2$$

$$\dashv\vdash F(P_1 \otimes UFP_2) \dashv\vdash P_1 \otimes FUFP_2$$

$$\dashv\vdash P_1 \otimes FP_2 \dashv\vdash F(P_1 \otimes P_2)$$

The first step in this computation also tacitly uses (in addition to the induction hypothesis) the functoriality of $\otimes, F, U$. ■

4 The Token-Passing Adjunction

By judicious choice of adjunction, the correctness of focusing proof search is a corollary of the above result. Take an atom q^+ that is fresh in the sense that it is disallowed from appearing in any other proposition, and include the following rules to define F and U.

$$\frac{\Gamma \vdash A}{\Gamma, q^+ \vdash FA} \qquad \frac{\Delta, A, q^+ \vdash C}{\Delta, FA \vdash C}$$

$$\frac{\Gamma, q^+ \vdash B}{\Gamma \vdash UB} \qquad \frac{\Delta, B \vdash C}{\Delta, UB, q^+ \vdash C}$$

Note that FA is essentially $A \otimes q^+$ and UB is essentially $q^+ \multimap B$, (making them linear, atomic versions of the adjunction behind the store monad) but they require q^+ to immediately be in the context during their synchronous decomposition steps. The fact that q^+ is labelled as a positive atom is not essential, since its interactions are completely governed by F and U, but it is appropriately suggestive to think of it as positive, because that polarity assignment intuitively justifies coalescing it with $\otimes$ and the domain of $\multimap$.

Since technically we are no longer working in just ordinary linear logic, it is worth noting that the logic with this particular F and U is still globally sound and complete:

Lemma 4.1 (Cut Admissibility) *If* $\Gamma \vdash A$ *and* $\Delta, A \vdash C$ *then* $\Gamma, \Delta \vdash C$.

Lemma 4.2 (Identity) $A \vdash A$ *for any* A.

Extending the proofs of these theorems to cover F and U is straightforward.

5 Focusing

Fact 5.1 *F is left invertible, and U is right invertible.*

We write $\vdash_i$ to indicate inverting proofs. The completeness of inverting proofs follows directly from the definition of what it means to be an invertible rule, and we have easily therefore that $\Gamma \vdash A \Leftrightarrow \Gamma \vdash_i A$. Write $\vdash_f$ to indicate focusing proofs.

The rough shape of what we can show is this:

$$\vdash_f A \quad \Leftrightarrow \quad \vdash_i A^\bullet \quad \Leftrightarrow \quad \vdash A^\bullet \quad \Leftrightarrow \quad \vdash A^\circ \quad \Leftrightarrow \quad \vdash_i A^\circ \quad \Leftrightarrow \quad \vdash A^-$$

That is, a focused proof is essentially the same thing as (in fact there is an isomorphism of sets of proofs) an inverting proof in the system where $\uparrow, \downarrow$ are interpreted as F, U, which exists iff there is a (not necessarily inverting) proof in the system, which exists iff there is a proof of the defocalization, which exists iff there is an inverting proof of the defocalization, which is essentially the same thing as a proof of A with all the shifts removed, again obtaining an isomorphsm of proofs as long as A prior to defocalization has no 'extra' unnecessary shift pairs.

To be more precise, define A^n and A^p by

$$\begin{array}{ll} P^n = \uparrow P & N^n = N \\ P^p = P & N^p = \downarrow N \end{array}$$

All the above operations lift to contexts in the obvious way.

Theorem 5.2 (Correctness of Focusing) *The following are equivalent:*

1. $\Delta^n \vdash_f A^p$
2. $U(\Delta^n)^\bullet, q^+ \vdash_i F(A^p)^\bullet$
3. $U(\Delta^n)^\bullet, q^+ \vdash F(A^p)^\bullet$
4. $U(\Delta^n)^\circ, q^+ \vdash F(A^p)^\circ$
5. $U(\Delta^n)^\circ, q^+ \vdash_i F(A^p)^\circ$
6. $\Delta^- \vdash A^-$

Proof The central equivalence (3) $\Leftrightarrow$ (4) follows immediately from Lemma 3.1. We get the next outermost equivalences (2) $\Leftrightarrow$ (3) and (4) $\Leftrightarrow$ (5) from the correctness of inverting proof search.

For the the outer equivalences (1) $\Leftrightarrow$ (2) and (5) $\Leftrightarrow$ (6), we need only check that inverting proof search on $A^\bullet$ is essentially focusing proof search on A, and that inverting proof search on A° is essentially ordinary proof search on A. The proofs of these facts are routine inductions that match up proof steps one-for-one, so we will merely consider an illustrative example.

Let $A = (\downarrow\uparrow P_1) \multimap (\uparrow\downarrow N_1 \mathbin{\&} \uparrow(\downarrow N_2 \oplus \downarrow N_3))$ and suppose Δ is full of negative propositions, and consider the sequent

$$\Delta, A \vdash P$$

If we focus on A we should be required to decompose both the $\multimap$ and the $\&$ together, and subsequently be required to decompose the $\oplus$ because it is invertible on the left. Under translation, this becomes

$$U\Delta^\bullet, UA^\bullet, q^+ \vdash FP^\bullet$$

where $A^\bullet = (UFP_1^\bullet) \multimap (FUN_1^\bullet \mathbin{\&} F(UN_2^\bullet \oplus UN_3^\bullet))$, If we decide to decompose $UA^\bullet$, then we consume the token q^+ are are left with the proof goal

$$U\Delta^\bullet, A^\bullet \vdash FP^\bullet$$

and we have no choice but to continue decomposing $A^\bullet$, because every other decomposition available requires q^+ to be present. So we proceed (imitating whatever resource distribution the focusing proof makes) with

$$\frac{U\Delta_1 \vdash UFP_1^\bullet \qquad U\Delta_2^\bullet, FUN_1^\bullet \mathbin{\&} F(UN_2^\bullet \oplus UN_3^\bullet) \vdash FP^\bullet}{U\Delta_1^\bullet, U\Delta_2^\bullet, A^\bullet \vdash FP^\bullet}$$

In the left branch we are compelled by the requirement to make an inverting proof to decompose the U on the right. In the right branch we are again forced by lack of q^+ to decompose the & — suppose the focusing proof takes the right conjunct, and we therefore imitate it. We arrive at the partial derivation

$$\dfrac{\dfrac{U\Delta_1, q^+ \vdash FP_1^\bullet}{U\Delta_1 \vdash UFP_1^\bullet} \qquad \dfrac{U\Delta_2^\bullet, F(UN_2^\bullet \oplus UN_3^\bullet) \vdash FP^\bullet}{U\Delta_2^\bullet, FUN_1^\bullet \mathbin{\&} F(UN_2^\bullet \oplus UN_3^\bullet) \vdash FP^\bullet}}{U\Delta_1^\bullet, U\Delta_2^\bullet, A^\bullet \vdash FP^\bullet}$$

The left branch already lines up correctly with the induction hypotheses and corresponds to a focusing proof of $\Delta_1 \vdash P$, and in the right branch we must decompose the invertible F and $\oplus$ to arrive at the two goals

$$U\Delta_2^\bullet, UN_i^\bullet, q^+ \vdash FP^\bullet$$

for $i \in \{2, 3\}$, which correspond by induction hypothesis to $\Delta_2, N_i \vdash_f P$. ■

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