Cyclic Dependent Types

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1 Introduction

There is no real obstacle to having dependent types where the dependency graph of a context (i.e. which types refer to which variables) has cycles. In order to make it sensible it seems necessary to require that everything we even mention is simply-typed, but this is likely good hygiene anyway, and simplifies many definitions and proofs.

2 Syntax

Define simple types and typed vectors of other things $X$ by

<table>
<thead>
<tr>
<th>Simple Types</th>
<th>$\tau$</th>
<th>::=</th>
<th>$(\tau_1, \ldots, \tau_n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Typed Vectors</td>
<td>$\bar{X}_{(\tau_1, \ldots, \tau_n)}$</td>
<td>::=</td>
<td>$(X_{\tau_1}, \ldots, X_{\tau_n})$</td>
</tr>
</tbody>
</table>

Assume variables $x_{\tau}$ are intrinsically simply typed. Other syntactic constructs are also intrinsically typed:

<table>
<thead>
<tr>
<th>Normal Terms</th>
<th>$M_{\tau}$</th>
<th>::=</th>
<th>$\lambda x_{\tau}. R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Classifiers</td>
<td>$V_{\tau}$</td>
<td>::=</td>
<td>$\Pi \Psi_{\tau}. v$</td>
</tr>
<tr>
<td>Atomic Terms</td>
<td>$R_{\tau}$</td>
<td>::=</td>
<td>$x_{\tau}[M_{\tau}]$</td>
</tr>
<tr>
<td>Base Classifiers</td>
<td>$v_{\tau}$</td>
<td>::=</td>
<td>$R_{\tau} \mid \text{type}$</td>
</tr>
<tr>
<td>Contexts</td>
<td>$\Psi_{\tau}$</td>
<td>::=</td>
<td>$\bar{x}<em>{\tau}, \bar{V}</em>{\tau}$</td>
</tr>
</tbody>
</table>

The judgments are:

| $\Gamma \vdash M_{\tau} \leftarrow V_{\tau}$ | $M$ checks at classifier $V$ |
| $\Gamma \vdash M_{\tau} \leftarrow \Psi$ | $M$ checks at context $\Psi$ |
| $\Gamma \vdash V_{\tau} \leftarrow \text{class}$ | $V$ is a well-formed classifier |
| $\Gamma \vdash \bar{V}_{\tau} \leftarrow \text{class}$ | $\bar{V}$ are well-formed classifiers |
| $\Gamma \vdash R \Rightarrow v$ | $R$ synthesizes classifier $v$ |
| $\Gamma \vdash v \Rightarrow \text{class}$ | $v$ is a well-formed base classifier |
| $\Gamma \vdash \Psi_{\tau} \text{ ctx}$ | $\Psi$ is a well-formed context |

$\Psi$ means just the variable vector from $\Psi$. $\Gamma, \Psi$ means concatenate the variable vectors and the type vectors of $\Gamma$ and $\Psi$. The judgments are defined by:
\[(M) \quad \frac{\Gamma, \Psi \vdash R \Rightarrow v' \quad v = v'}{\Gamma \vdash \lambda \Psi.R \leftarrow \Pi \Psi.v} \quad (V) \quad \frac{\Gamma, \Psi \vdash \varepsilon}{\Gamma \vdash \Pi \Psi.v \leftarrow \text{class}} \quad (R) \quad \frac{x : \Pi \Psi.v \in \Gamma \quad \Gamma \vdash \bar{M} \leftarrow \Psi \{\bar{M}/\hat{\Psi}\}}{\Gamma \vdash x[M] \Rightarrow \nu\{\bar{M}/\hat{\Psi}\}} \quad (v) \quad \frac{\Gamma \vdash R \Rightarrow \text{type}}{\Gamma \vdash R \Rightarrow \text{class}} \quad (\Psi) \quad \frac{\Gamma \vdash (\bar{x}.\bar{V}) \leftarrow \text{class}}{\Gamma \vdash (\bar{x}.\bar{V}) \text{ ctx}}\]

Substitution is written \(\{M/x\} = \{\bar{M}/\bar{x}\}\). We write \((M/x) \in \{\bar{M}/\bar{x}\}\) when, for some \(n\), we have \(M = \bar{M}_n\) and \(x = \bar{x}_n\). The behavior of substitution is all boring congruences except for the variable case. Abbreviating \(\theta = \{\bar{M}/\bar{x}\}\),

\[(x[\bar{N}]\theta) = \begin{cases} R[\bar{N}\theta/\bar{y}] & \text{if } (\lambda \bar{y}.R/x) \in \theta; \\ x[\bar{N}\theta] & \text{otherwise.} \end{cases}\]

3 Results

We prove the usual results; that substitution and identity properties hold. In preparation for the substitution property, we show that substitutions commute properly. In preparation for the identity property, we show that \(\eta\)-expansions are two-sided units with respect to substitution.

3.1 Substitution

Lemma 3.1 If \(FV(X) \cap \bar{x} = \emptyset\), then \(X \{\bar{M}/\bar{x}\} = X\).

Proof Straightforward induction. ■

Let \(J\) stand for an arbitrary judgment of the system.

Lemma 3.2 (Weakening) If \(\Gamma \vdash J\), then \(\Gamma, \Gamma' \vdash J\).

Proof Straightforward induction. ■

Abbreviate \(\theta = \{\bar{M}/\bar{x}\}\).

Lemma 3.3 (Interchange) If \(FV(\bar{M}) \cap \bar{y} = \emptyset\), then \(X \{\bar{N}/\bar{y}\}\theta = X\theta(\bar{N}\theta/\bar{y})\)

Proof By induction on first the unordered pair of the simple types of \(\bar{x}, \bar{y}\), and subsequently \(X\). For all the homomorphism cases, it’s just \(X\) that gets smaller. This includes the case of \(X = z[\bar{P}]\) where \(z\) is in neither \(\bar{x}\) nor \(\bar{y}\). The interesting cases are when \(X = z[\bar{P}]\) and

2
• \( z \in \bar{x} \) and \((\lambda \bar{w}.R/z) \in \{\bar{M}/\bar{x}\}\). In this case we reason that

\[
\begin{align*}
z[P]{\bar{N}/\bar{y}}\theta \\
= z[P]{\bar{N}/\bar{y}}\theta \\
= R[P]{\bar{N}/\bar{y}}\theta/\bar{w} \\
= R[P\theta]{\bar{N}/\bar{y}}\theta/\bar{w} & \quad \text{i.h. on } \bar{P} < z[\bar{P}] \\
= R[P\theta]{\bar{N}/\bar{y}}\theta/\bar{w} & \quad \text{i.h. on } (\bar{w}, \bar{y}) < (\bar{y}, \bar{x}) \\
= z[P\theta]{\bar{N}/\bar{y}} \\
= z[P\theta]{\bar{N}/\bar{y}} \quad \text{Lemma 3.1}
\end{align*}
\]

To justify the second induction hypothesis appeal, we need \( \text{FV}(\bar{N} \theta) \cap \bar{w} = \emptyset \), but this is true because the variables \( \bar{w} \) are bound inside \( \bar{M} \).

• \( z \in \bar{y} \) and \((\lambda \bar{w}.R/z) \in \{\bar{N}/\bar{y}\}\). In this case we reason that

\[
\begin{align*}
z[P]{\bar{N}/\bar{y}}\theta \\
= R[P\theta]{\bar{N}/\bar{y}}\theta/\bar{w} \\
= R[P\theta]{\bar{N}/\bar{y}}\theta/\bar{w} & \quad \text{i.h. on } \bar{P} < z[\bar{P}] \\
= z[P\theta]{\bar{N}/\bar{y}} \quad \text{Lemma 3.1}
\end{align*}
\]

To justify the first induction hypothesis appeal, we need \( \text{FV}(\bar{M} \cap \bar{w} = \emptyset \), but this is true because the variables \( \bar{w} \) are bound inside \( \bar{N} \).

Abbreviate \( \theta = \{\bar{M}/\hat{\Gamma}\} \).

**Lemma 3.4** If \( \Delta, \Gamma \vdash J \) and \( \Delta \theta \vdash \bar{M} \leftarrow \Gamma \theta \), then \( \Delta \theta \vdash J \theta \).

**Proof** By induction on first the simple type of \( \Gamma \) and subsequently the derivation of \( J \). The interesting case is:

\[
\begin{align*}
\text{Case: } & D' \\
\text{D} = x : \Pi \Psi . v \in \Gamma & \Delta, \Gamma \vdash \bar{N} \leftarrow \Psi \{\bar{N}/\hat{\Psi}\} \\
\Delta, \Gamma \vdash x[\bar{N}] \Rightarrow v\{\bar{N}/\hat{\Psi}\}
\end{align*}
\]

Use the induction hypothesis on \( D' \) to see \( \Delta \theta \vdash \bar{N} \theta \leftarrow \Psi \{\bar{N}/\hat{\Psi}\}\). By simple types we must have some \( \lambda \Psi . R/x \in \theta \), and by picking apart the typing of \( \bar{M} \) we must have had \( \Delta \theta \vdash \lambda \hat{\Psi}. R \leftarrow \Pi \Psi \cdot v \theta \), so by inversion \( \Delta \theta, \Psi \theta \vdash R \Rightarrow v \theta \).

We claim we’re in a position to apply the induction hypothesis. Why? The substitution is \( \{\bar{N} \theta/\hat{\Psi}\} \), which substitutes for a smaller simple type
than \( \theta \). None of \( \hat{\Psi} \) were bound in \( \Delta \) so we don’t need to worry about the substitution in the second premise’s context left of the turnstile. On the right of the second premise the context must be \( \Psi\{N/\hat{\Psi}\} \), which is equal to what we have, \( \Psi\{N/\hat{\Psi}\}\theta \), by lemma, noting that \( \Psi \) were too recently bound to occur in \( \bar{M} \).

So out of the induction hypothesis comes \( \Delta \theta \vdash R\{N\theta/\hat{\Psi}\} \Rightarrow v\{N\theta/\hat{\Psi}\} \). After one more application of the above lemma, we have

\[
\Delta \theta \vdash R\{N\theta/\hat{\Psi}\} \Rightarrow v\{N\theta/\hat{\Psi}\} \theta
\]

as required.

\[ \blacksquare \]

### 3.2 Identity

Eta-expansion is defined on variables and variable vectors (yielding terms and term vectors) by

\[
\eta(x) = \lambda \bar{y}.x[\eta(\bar{y})]
\]

\[
\eta(x_1, \ldots, x_n) = \eta(x_1), \ldots, \eta(x_n)
\]

**Lemma 3.5** (Unit Laws for \( \eta \)-expansion)

1. \( X\{\eta(\bar{x})/\bar{x}\} = X \)
2. \( \eta(\bar{x})\{M/\bar{x}\} = M \)

**Proof** By induction on \( \tau \) and \( X \) or \( \bar{M} \). \[ \blacksquare \]

**Lemma 3.6** (Identity)

1. If \( x : V \in \Gamma \), then \( \Gamma \vdash \eta(x) \leftarrow V \).
2. If \( \bar{x} : \bar{V} \subseteq \Gamma \), then \( \Gamma \vdash \eta(\bar{x}) \leftarrow \bar{V} \).

**Proof** By induction. In the first part, form the derivation

\[
\begin{aligned}
D' \\
\frac{x : \Pi \Psi.v \in \Gamma \quad \Gamma, \Psi \vdash \eta(\bar{y}) \leftarrow \Psi \{\eta(\bar{y})/\hat{\Psi}\}}{\Gamma, \Psi \vdash \eta(\bar{y}) \leftarrow \Psi \{\eta(\bar{y})/\hat{\Psi}\} \eta id} \\
\frac{\Gamma \vdash \lambda\bar{y}.x[\eta(\bar{y})] \Rightarrow \Pi \Psi.v\{\eta(\bar{y})/\hat{\Psi}\}}{\Gamma \vdash \lambda\bar{y}.x[\eta(\bar{y})] \Rightarrow \Pi \Psi.v} \eta id
\end{aligned}
\]

from \( D' \) obtained from the induction hypothesis, using the above lemma at steps marked \( \eta id \). \[ \blacksquare \]