Cyclic Dependent Types

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1 Introduction

There is no real obstacle to having dependent types where the dependency graph of a context (i.e. which types refer to which variables) has cycles. In order to make it sensible it seems necessary to require that everything we even mention is simply-typed, but this is likely good hygiene anyway, and simplifies many definitions and proofs.

2 Syntax

Define simple types and typed vectors of other things X by

Simple Types	τ	$::=$	$(\tau_1, \dots, \tau_n)$
Typed Vectors	$\bar{X}_{(\tau_1,\dots,\tau_n)}$	$::=$	$(X_{\tau_1}, \dots, X_{\tau_n})$

Assume variables x_τ are intrinsically simply typed. Other syntactic constructs are also intrinsically typed:

Normal Terms	M_τ	$::=$	$\lambda\bar{x}_\tau.R$
Classifiers	V_τ	$::=$	$\Pi\Psi_\tau.v$
Atomic Terms	R	$::=$	$x_\tau[\bar{M}_\tau]$
Base Classifiers	v	$::=$	$R \mid \mathrm{type}$
Contexts	Ψ_τ	$::=$	$\bar{x}_\tau.\bar{V}_\tau$

The judgments are:

$\Gamma \vdash M_\tau \Leftarrow V_\tau$	M checks at classifier V
$\Gamma \vdash \bar{M}_\tau \Leftarrow \Psi$	$\bar{M}$ checks at context Ψ
$\Gamma \vdash V_\tau \Leftarrow \mathrm{class}$	V is a well-formed classifier
$\Gamma \vdash \bar{V}_\tau \Leftarrow \overline{\mathrm{class}}$	$\bar{V}$ are well-formed classifiers
$\Gamma \vdash R \Rightarrow v$	R synthesizes classifier v
$\Gamma \vdash v \Rightarrow \mathrm{class}$	v is a well-formed base classifier
$\Gamma \vdash \Psi_\tau\ \mathrm{ctx}$	Ψ is a well-formed context

$\hat{\Psi}$ means just the variable vector from Ψ. Γ, Ψ means concatenate the variable vectors and the type vectors of Γ and Ψ. The judgments are defined by:

$$(M) \quad \frac{\Gamma, \Psi \vdash R \Rightarrow v' \qquad v = v'}{\Gamma \vdash \lambda\hat{\Psi}.R \Leftarrow \Pi\Psi.v} \qquad (\bar{M}) \quad \frac{\Gamma \vdash M_i \Leftarrow V_i}{\Gamma \vdash \bar{M} \Leftarrow \bar{x}.\bar{V}}$$

$$(V) \quad \frac{\Gamma \vdash \Psi \text{ ctx} \qquad \Gamma, \Psi \vdash v \Rightarrow \text{class}}{\Gamma \vdash \Pi\Psi.v \Leftarrow \text{class}} \qquad (\bar{V}) \quad \frac{\Gamma \vdash V_i \Leftarrow \text{class}}{\Gamma \vdash \bar{V} \Leftarrow \overline{\text{class}}}$$

$$(R) \quad \frac{x : \Pi\Psi.v \in \Gamma \qquad \Gamma \vdash \bar{M} \Leftarrow \Psi\{\bar{M}/\hat{\Psi}\}}{\Gamma \vdash x[\bar{M}] \Rightarrow v\{\bar{M}/\hat{\Psi}\}}$$

$$(v) \quad \frac{\Gamma \vdash R \Rightarrow \text{type}}{\Gamma \vdash R \Rightarrow \text{class}} \quad \frac{}{\Gamma \vdash \text{type} \Rightarrow \text{class}} \qquad (\Psi) \quad \frac{\Gamma, (\bar{x}.\bar{V}) \vdash \bar{V} \Leftarrow \overline{\text{class}}}{\Gamma \vdash (\bar{x}.\bar{V}) \text{ ctx}}$$

Substitution is written $\{\bar{M}_\tau/\bar{x}_\tau\}$. We write $(M/x) \in \{\bar{M}/\bar{x}\}$ when, for some n, we have $M = M_n$ and $x = x_n$. The behavior of substitution is all boring congruences except for the variable case. Abbreviating $\theta = \{\bar{M}/\bar{x}\}$,

$$(x[\bar{N}])\theta = \begin{cases} R\{\bar{N}\theta/\bar{y}\} & \text{if } (\lambda\bar{y}.R/x) \in \theta; \\ x[\bar{N}\theta] & \text{otherwise.} \end{cases}$$

3 Results

We prove the usual results; that substitution and identity properties hold. In preparation for the substitution property, we show that substitutions commute properly. In preparation for the identity property, we show that η-expansions are two-sided units with respect to substitution.

3.1 Substitution

Lemma 3.1 *If $FV(X) \cap \bar{x} = \emptyset$, then $X\{\bar{M}/\bar{x}\} = X$.*

Proof Straightforward induction. ■

Let J stand for an arbitrary judgment of the system.

Lemma 3.2 (Weakening) *If $\Gamma \vdash J$, then $\Gamma, \Gamma' \vdash J$.*

Proof Straightforward induction. ■

Abbreviate $\theta = \{\bar{M}/\bar{x}\}$.

Lemma 3.3 (Interchange) *If $FV(\bar{M}) \cap \bar{y} = \emptyset$, then $X\{\bar{N}/\bar{y}\}\theta = X\theta\{\bar{N}\theta/\bar{y}\}$*

Proof By induction on first the undordered pair of the simple types of $\bar{x}, \bar{y}$, and subsequently X. For all the homomorphism cases, it's just X that gets smaller. This includes the case of $X = z[\bar{P}]$ where z is in neither $\bar{x}$ nor $\bar{y}$. The interesting cases are when $X = z[\bar{P}]$ and

- $z \in \bar{x}$ and $(\lambda\bar{w}.R/z) \in \{\bar{M}/\bar{x}\}$. In this case we reason that

  $$\begin{array}{lr} z[\bar{P}]\{\bar{N}/\bar{y}\}\theta & \\ = z[\bar{P}\{\bar{N}/\bar{y}\}]\theta & \\ = R\{\bar{P}\{\bar{N}/\bar{y}\}\theta/\bar{w}\} & \\ = R\{\bar{P}\theta\{\bar{N}\theta/\bar{y}\}/\bar{w}\} & \text{i.h. on } \bar{P} < z[\bar{P}] \\ = R\{\bar{N}\theta/\bar{y}\}\{\bar{P}\theta\{\bar{N}\theta/\bar{y}\}/\bar{w}\} & \text{Lemma 3.1} \\ = R\{\bar{P}\theta/\bar{w}\}\{\bar{N}\theta/\bar{y}\} & \text{i.h. on } (\bar{w},\bar{y}) < (\bar{y},\bar{x}) \\ = z[\bar{P}]\theta\{\bar{N}\theta/\bar{y}\} & \end{array}$$

  To justify the second induction hypothesis appeal, we need $FV(\bar{N}\theta)\cap\bar{w} = \emptyset$, but this is true because the variables $\bar{w}$ are bound inside $\bar{M}$.

- $z \in \bar{y}$ and $(\lambda\bar{w}.R/z) \in \{\bar{N}/\bar{y}\}$. In this case we reason that

  $$\begin{array}{lr} z[\bar{P}]\{\bar{N}/\bar{y}\}\theta & \\ = R\{\bar{P}\{\bar{N}/\bar{y}\}/\bar{w}\}\theta & \\ = R\theta\{\bar{P}\{\bar{N}/\bar{y}\}\theta/\bar{w}\} & \text{i.h. on } (\bar{w},\bar{x}) < (\bar{y},\bar{x}) \\ = R\theta\{\bar{P}\theta\{\bar{N}\theta/\bar{y}\}/\bar{w}\} & \text{i.h. on } \bar{P} < z[\bar{P}] \\ = R\{\bar{P}\theta\{\bar{N}\theta/\bar{y}\}/\bar{w}\} & \text{Lemma 3.1} \\ = z[\bar{P}\theta]\{\bar{N}\theta/\bar{y}\} & \\ = z[\bar{P}]\theta\{\bar{N}\theta/\bar{y}\} & \end{array}$$

  To justify the first induction hypothesis appeal, we need $FV(\bar{M})\cap\bar{w} = \emptyset$, but this is true because the variables $\bar{w}$ are bound inside $\bar{N}$.

■

Abbreviate $\theta = \{\bar{M}/\hat{\Gamma}\}$.

Lemma 3.4 *If* $\Delta,\Gamma \vdash J$ *and* $\Delta\theta \vdash \bar{M} \Leftarrow \Gamma\theta$, *then* $\Delta\theta \vdash J\theta$.

Proof By induction on first the simple type of Γ and subsequently the derivation of J. The interesting case is:

Case:

$$\mathcal{D} = \cfrac{x : \Pi\Psi.v \in \Gamma \qquad \overset{\mathcal{D}'}{\Delta,\Gamma \vdash \bar{N} \Leftarrow \Psi\{\bar{N}/\hat{\Psi}\}}}{\Delta,\Gamma \vdash x[\bar{N}] \Rightarrow v\{\bar{N}/\hat{\Psi}\}}$$

Use the induction hypothesis on $\mathcal{D}'$ to see $\Delta\theta \vdash \bar{N}\theta \Leftarrow \Psi\{\bar{N}/\hat{\Psi}\}\theta$. By simple types we must have some $\lambda\hat{\Psi}.R/x \in \theta$, and by picking apart the typing of $\bar{M}$ we must have had $\Delta\theta \vdash \lambda\hat{\Psi}.R \Leftarrow \Pi\Psi\theta.v\theta$, so by inversion $\Delta\theta, \Psi\theta \vdash R \Rightarrow v\theta$.

We claim we're in a position to apply the induction hypothesis. Why? The substitution is $\{\bar{N}\theta/\hat{\Psi}\}$, which substitutes for a smaller simple type than θ. None of $\hat{\Psi}$ were bound in Δ so we don't need to worry about the substitution in the second premise's context left of the turnstile. On the right of the second premise the context must be $\Psi\theta\{\bar{N}\theta/\hat{\Psi}\}$, which is equal to what we have, $\Psi\{\bar{N}/\hat{\Psi}\}\theta$, by lemma, noting that Ψ were too recently bound to occur in $\bar{M}$.

So out of the induction hypothesis comes $\Delta\theta \vdash R\{\bar{N}\theta/\hat{\Psi}\} \Rightarrow v\theta\{\bar{N}\theta/\hat{\Psi}\}$. After one more application of the above lemma, we have

$$\Delta\theta \vdash R\{\bar{N}\theta/\hat{\Psi}\} \Rightarrow v\{\bar{N}/\hat{\Psi}\}\theta$$

as required.

■

3.2 Identity

Eta-expansion is defined on variables and variable vectors (yielding terms and term vectors) by

$$\eta(x_\tau) = \lambda\bar{y}_\tau.x[\eta(\bar{y}_\tau)]$$

$$\eta(x_1, \ldots, x_n) = \eta(x_1), \ldots, \eta(x_n)$$

Lemma 3.5 (Unit Laws for η-expansion)

1. $X\{\eta(\bar{x}_\tau)/\bar{x}_\tau\} = X$
2. $\eta(\bar{x}_\tau)\{\bar{M}_\tau/\bar{x}_\tau\} = \bar{M}_\tau$

Proof By induction on τ and X or $\bar{M}$. ■

Lemma 3.6 (Identity)

1. *If* $x : V \in \Gamma$, *then* $\Gamma \vdash \eta(x) \Leftarrow V$.
2. *If* $\bar{x} : \bar{V} \subseteq \Gamma$, *then* $\Gamma \vdash \eta(\bar{x}) \Leftarrow \bar{V}$.

Proof By induction. In the first part, form the derivation

$$\dfrac{\dfrac{\dfrac{x : \Pi\Psi.v \in \Gamma \qquad \dfrac{\begin{matrix}\mathcal{D}' \\ \Gamma, \Psi \vdash \eta(\bar{y}) \Leftarrow \Psi\end{matrix}}{\Gamma, \Psi \vdash \eta(\bar{y}) \Leftarrow \Psi\{\eta(\bar{y})/\hat{\Psi}\}}\,\eta\mathsf{id}}{\Gamma, \Psi \vdash x[\bar{M}] \Rightarrow v\{\eta(\bar{y})/\hat{\Psi}\}}}{\Gamma \vdash \lambda\bar{y}.x[\eta(\bar{y})] \Rightarrow \Pi\Psi.v\{\eta(\bar{y})/\hat{\Psi}\}}}{\Gamma \vdash \lambda\bar{y}.x[\eta(\bar{y})] \Rightarrow \Pi\Psi.v}\,\eta\mathsf{id}$$

from $\mathcal{D}'$ obtained from the induction hypothesis, using the above lemma at steps marked $\eta\mathsf{id}$. ■