The Identity Theorem in Hereditary Spine Form

Jason Reed

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Terms

\[ M ::= x \cdot S | \lambda x. S \]

Spines

\[ S ::= () | (M; S) \]

Syntactic Objects

\[ X ::= M \cdot S \]

Eta-expansion:

\[ \text{ex}_\alpha (H \cdot S) = H \cdot S \]
\[ \text{ex}_{A \rightarrow B} (H \cdot S) = \lambda y. \text{ex}_B (H \cdot (S \cdot \text{ex}_A (x \cdot ()))) \]

Substitution and reduction are as usual, abbreviating \( \sigma = [M/x]^A \):

\[ \sigma (\lambda y. N) = \lambda y. \sigma N \]
\[ \sigma (y \cdot S) = y \cdot \sigma S \]
\[ \sigma (x \cdot S) = [M \mid \sigma S] \]
\[ \sigma () = () \]
\[ \sigma (N; S) = (\sigma N; \sigma S) \]
\[ [\lambda x. N \mid (M; S)]^{A \rightarrow B} = [[M/x]^A N \mid S]^B \]
\[ [x \cdot S \mid ()]^\sigma = x \cdot S \]
\[ [M \mid ()]^A = M \]

In all the following lemmas, \( M \Rightarrow N \) means that if \( M \) is defined, then \( N \) is defined, and \( M = N \).

0.1 Lemmas

**Lemma 0.1** \([M \mid (S; S')]^B \Rightarrow [[M \mid S]^B_1 \mid S']^B_2 \) for some \( B_1, B_2 \).

**Proof** Induction on \( S \).

Case: \( S = () \). Immediate by definition of reduction. Pick \( B_1 = a \) and \( B_2 = B \).

Case: \( S = (M_0; S_0) \). Then \( M \) is of the form \( \lambda y. N \), and \( B \) must be of the form \( B_L \rightarrow B_R \).

\[
[M \mid (S; S')]^B = [\lambda y. N \mid (M_0; S_0; S')]^B \\
= [[M_0/y]^B_L \cdot N \mid (S_0; S')]^B_R \\
= [[[M_0/y]^B_L \cdot N \mid S_0]^B_{R_1} \mid S']^B_{R_2} \\
= \text{by i.h.} \\
= [[\lambda y. N \mid (M_0; S_0)]^{B_L \rightarrow B_{R_1}} \mid S']^{B_{R_2}} \\
= [[M \mid S]^B_{L \rightarrow B_{R_1}} \mid S']^{B_{R_2}}
\]
Lemma 0.2 \([M/x]^A\text{ex}_B(y \cdot S) \Rightarrow \text{ex}_B(y \cdot [M/x]^A S)\).

**Proof** By induction on \(B\).

Case: \(B = a\). Immediate by definition of \(\text{ex}\) and substitution.

Case: \(B = B_1 \rightarrow B_2\). Then

\[
[M/x]^A\text{ex}_{B_1 \rightarrow B_2}(y \cdot S) = [M/x]^A \lambda z. \text{ex}_{B_1}(y \cdot (S; \text{ex}_{B_1}(z \cdot ())))
\]

\[
= \lambda z.[M/x]^A \text{ex}_{B_2}(y \cdot (S; \text{ex}_{B_1}(z \cdot ())))
\]

\[
= \lambda z. \text{ex}_{B_2}(y \cdot ([M/x]^A S; [M/x]^A \text{ex}_{B_1}(z \cdot ())))
\]

\[
= \lambda z. \text{ex}_{B_2}(y \cdot ([M/x]^A S; \text{ex}_{B_1}([M/x]^A(z \cdot ()))))
\]

\[
= \lambda z. \text{ex}_{B_2}(y \cdot ([M/x]^A S; \text{ex}_{B_1}(z \cdot ())))
\]

\[
= \text{ex}_{B_1 \rightarrow B_2}(y \cdot ([M/x]^A S))
\]

Lemma 0.3

1. \([\text{ex}_A(x \cdot ())/y]^B X \Rightarrow [x/y]X\)
2. \([\text{ex}_A(x \cdot S) \mid S']^B \Rightarrow x \cdot (S; S')\)
3. \([M/x]^B \text{ex}_A(x \cdot S) \Rightarrow [M][S']^B\), if \(x \not\in \text{FV}(S)\), for some \(B'\).

**Proof** By lexicographic induction on \(A\), the case 1–3, and the object \(X\).

1. Split cases on the structure of \(X\). The reasoning is straightforward except when \(X = y \cdot S\). Then we must show, by the definition of substitution,

   \[
   \text{ex}_A(x \cdot ()) \mid [\text{ex}_A(x \cdot ())/y]^B S \Rightarrow x \cdot ([x/y]S)
   \]

   We get \([\text{ex}_A(x \cdot ())/y]^B S = [x/y]S\) from the i.h. part 1, on the smaller expression \(S\). Then appeal to the i.h. part 2 on the same type \(A\) to see that

   \[
   [\text{ex}_A(x \cdot ())] \mid [x/y]S^B \Rightarrow x \cdot ([x/y]S)
   \]

2. Split cases on \(A\).

   Case: \(A = a\). Immediate from definitions. Note that \(S'\) must be () and \(B\) must be a base type for the left hand side to be defined.

   Case: \(A = A_1 \rightarrow A_2\). \(S'\) must be of the form \((M_0; S_0)\), and \(B\) must be of the form \(B_1 \rightarrow B_2\).
3. Split cases on $A$.

Case: $A = a$. Immediate from assumption and definitions.

Case: $A = A_1 \to A_2$. Note first of all that for any $N, C$ that

$$\lambda y. [N \| (\text{ex}_{A_1}(y \cdot ()))]^C \to N$$

This is because $N$ must be of the form $\lambda z. N_0$ and $C$ of the form $C_1 \to C_2$ for the left-hand side to be defined, in which case we have

$$\lambda y. [\lambda z. N_0 \| (\text{ex}_{A_1}(y \cdot ()))]^{C_1 \to C_2}$$

$$= \lambda y. [\text{ex}_{A_1}(y \cdot ())/z]^{C_1} N_0 || ()))^{C_2}$$

$$= \lambda y. [\text{ex}_{A_1}(y \cdot ())/z]^{C_1} N_0$$

$$= \lambda y. y/z] N_0$$

$$= N$$

by i.h. 1 on $A_1$

by $\alpha$-equivalence

Having made this observation, compute

$$[M/x]^{B_{\text{ex}_{A_1} \to A_2}(x \cdot S)}$$

$$= [M/x]^{B_{\text{ex}_{A_1} \to A_2}(x \cdot S; \text{ex}_{A_1}(y \cdot ()))}$$

$$= \lambda y. [M/x]^{B_{\text{ex}_{A_1} \to A_2}(x \cdot (S; \text{ex}_{A_1}(y \cdot ()))}$$

$$= \lambda y. [M/(S; \text{ex}_{A_1}(y \cdot ()))]^{B'}$$

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$$= \lambda y. [M/(S; \text{ex}_{A_1}(y \cdot ()))]^{B'}$$

$$= [M/S]^{B_1}$$

by Lemma 0.1

by above observation

by i.h. 3 on $A_2$