# A Modular Proof of the Completeness of Focusing

Jason Reed

February 9, 2007

## 1 Language

$$\begin{array}{rl}
\text{Propositions } A ::= & p \mid F^+ \mid F^- \\
\text{Conclusions } C ::= & A \mid \mathbf{S}_-(n, A) \\
\text{Pos Props } F^+, G^+, H^+ ::= & F^+ \otimes F^+ \mid F^+ \oplus F^+ \mid \mathbf{S}^+(n, A) \mid \mathbf{d}^+ A \mid !F^- \\
\text{Neg Props } F^-, G^-, H^- ::= & F^+ \multimap F^- \mid F^- \mathbin{\&} F^- \mid \mathbf{S}^-(n, A) \mid \mathbf{d}^- A \\
\text{Unrestricted Contexts } \Gamma ::= & \cdot \mid \Gamma, F^- \ \mathsf{tr} \\
\text{Linear Contexts } \Delta ::= & \cdot \mid \Delta, A \ \mathsf{res} \mid \Delta, \mathbf{S}_+(n, A) \ \mathsf{res} \\
\text{Active Contexts } \Omega ::= & \cdot \mid \Omega, F^+ \ \mathsf{act} \mid \Omega, F^- \ \mathsf{act} \\
\text{Basic Context } \Xi ::= & \Gamma, \Delta \\
\text{Full Context } \Psi ::= & \Gamma, \Delta, \Omega
\end{array}$$

The default judgment, if a bare proposition appears without any further annotation, is the linear 'resource' truth $\mathsf{res}$. The other two judgments are (ordinary intuitionistic) truth $\mathsf{tr}$ and a new judgment $\mathsf{act}$, 'active truth'. Active hypotheses are ordered; if the conclusion is active, it must be decomposed before any hypothesis, and if any hypotheses are active, they must be decomposed in order, right-to-left, before any further decomposition of the conclusion or other non-active hypotheses.

If syntactic objects appear without any $\pm$ when they should have one, it means that either polarity can be consistently applied.

Truth assumptions $A \ \mathsf{tr} \in \Gamma$ are subject to weaking, contraction, and exchange, linear assumptions $A \ \mathsf{res} \in \Delta$ subject only to exchange, and active assumptions $A \ \mathsf{act} \in \Omega$ subject to no structural rules apart from the associativity implicit in presenting them as a list. A comma between contexts means multiset union, except on $\mathsf{tr}$ assumptions, for which it means union. It is an invariant that two contexts both containing $\mathsf{act}$ assumptions should never be so joined.

The two main judgment forms are

$$\Psi \vdash C \ \mathsf{res}$$

$$\Psi \vdash F^{\pm} \ \mathsf{act}$$

There is a third, derived judgment.

$$\Psi \vdash F^- \ \mathsf{tr}$$

which is defined to mean that $\Psi$ is of the form $\Gamma$ (i.e. contains only $\mathsf{tr}$ assumptions) and $\Gamma \vdash F^-\ \mathsf{res}$. Let $J$ stand for $A\ \mathsf{res}$ or $F^-\ \mathsf{tr}$.

The connectives $\mathbf{d}^+$ and $\mathbf{d}^-$ are 'deactivation' operators; I believe they roughly correspond to Girard's polarity shift operators $\downarrow$ and $\uparrow$, respectively. The difference is that I do not prohibit a decativation $\mathbf{d}^+$ between a bunch of positive connectives, and another bunch of also positive connectives; $\mathbf{d}^+$ doesn't necessarily mean a transition between positive and negative, but merely a place where consecutive positive decompositions may pause. What happens after those positive decompositions may be of either polarity.

In place of signed atomic propositions, there are signed (and essentially nullary) connectives $\mathbf{S}^+(n, A)$ and $\mathbf{S}^-(n, A)$. The natural number $n$ and proposition $A$ only serve as indices. That is, it is not possible for $\mathbf{S}^+(n, A) \vdash \mathbf{S}^+(n, B)$ to hold if $A$ and $B$ differ, even if $A \vdash B$. They can be equally well thought of as (signed) propositional atoms $p^+_{n,A}$ and $p^-n, A$. It just so happens to be useful to have the indices around as bookkeeping devices for the proof of completeness, and typographically less unpleasant to not have them as subscripts.

The way I treat signed propositions is slightly different than in prior work. The *proposition* $\mathbf{S}^+(n, A)$ actually does undergo an asynchronous decomposition on the left (as $\mathbf{S}^-$ does on the right) into a (subscripted) *judgmental form* $\mathbf{S}_+$ (respectively, $\mathbf{S}_-$) on the left (resp. right) of the turnstile. Elements $S_+(n, A)\ \mathsf{res}$ of $\Delta$, although they are not actually linear hypotheses, behave as if they were in the sense of not being subject to contraction, and being split across multiplicative splits.

## 2 Overview

The idea is that we can write down a complex connective like $(A \oplus B) \multimap (C \mathbin{\&} D)$ with no internal polarity changes in the present system as $(\mathbf{d}^+ A \oplus \mathbf{d}^+ B) \multimap (\mathbf{d}^- C \mathbin{\&} \mathbf{d}^- D)$. Whenever this proposition appears on either the left or right, we will force all connectives to be decomposed until we reach a $\mathbf{d}^\pm$ — the only way to decompose anything will be to first 'activate' it, which implies a commitment to decompose until 'deactivation'. We can show that the process of taking an ordinary sequent, and rewriting it so as to maximally coalesce unipolar segments of propositions, (that is, inserting as few $\mathbf{d}^\pm$s as possible) is complete. Now we have recovered focussed proofs as long as we decompose asynchronous complex connectives eagerly.

## 3 Proof Rules

Here are all the sequent rules, beginning with init:

$$\frac{}{\Gamma, p \vdash p}$$

These three structural rules are 'right activation', 'left activation', and 'copy':

$$\frac{\Xi \vdash F \ \mathsf{act}}{\Xi \vdash F} \qquad \frac{\Xi, F \ \mathsf{act} \vdash C}{\Xi, F \vdash C} \qquad \frac{\Xi, F^- \ \mathsf{tr}, F^- \ \mathsf{act} \vdash C}{\Xi, F^- \ \mathsf{tr} \vdash C}$$

These two are the left and right rules for the 'deactivation' operator $\mathbf{d}$ :

$$\frac{\Psi \vdash A}{\Psi \vdash \mathbf{d}^{\pm} A \ \mathsf{act}} \qquad \frac{\Psi, A \vdash C}{\Psi, \mathbf{d}^{\pm} A \ \mathsf{act} \vdash C}$$

All that's left is the usual connective left and right rules. Note that asynchronous decompositions allow a full $\Psi$, (which permits ordered assumptions) but synchronous decompositions restrict the remaining context to $\Xi$.

$$\frac{\Psi, F^+ \ \mathsf{act} \vdash F^- \ \mathsf{act}}{\Psi \vdash F^+ \multimap F^- \ \mathsf{act}} \qquad \frac{\Xi_1 \vdash F^+ \ \mathsf{act} \qquad \Xi_2, F^- \ \mathsf{act} \vdash C}{\Xi_1, \Xi_2, F^+ \multimap F^- \ \mathsf{act} \vdash C}$$

$$\frac{\Psi \vdash F_1^- \ \mathsf{act} \qquad \Psi \vdash F_2^- \ \mathsf{act}}{\Psi \vdash F_1^- \ \& \ F_2^- \ \mathsf{act}} \qquad \frac{\Xi, F_i^- \ \mathsf{act} \vdash C}{\Xi, F_1^- \ \& \ F_2^- \ \mathsf{act} \vdash C}$$

$$\frac{\Psi \vdash \mathbf{S}_-(n, A)}{\Psi \vdash \mathbf{S}^-(n, A) \ \mathsf{act}} \qquad \frac{}{\Gamma, \mathbf{S}^-(n, A) \ \mathsf{act} \vdash \mathbf{S}_-(n, A)}$$

$$\frac{\Psi, F_1^+ \ \mathsf{act} \vdash C \qquad \Psi, F_2^+ \ \mathsf{act} \vdash C}{\Psi, F_1^+ \oplus F_2^+ \ \mathsf{act} \vdash C} \qquad \frac{\Xi \vdash F_i^+ \ \mathsf{act}}{\Xi \vdash F_1^+ \oplus F_2^+ \ \mathsf{act}}$$

$$\frac{\Psi, F_1^+ \ \mathsf{act}, F_2^+ \ \mathsf{act} \vdash C}{\Psi, F_1^+ \otimes F_2^+ \ \mathsf{act} \vdash C} \qquad \frac{\Xi_1 \vdash F_1^+ \ \mathsf{act} \qquad \Xi_2 \vdash F_2^+ \ \mathsf{act}}{\Xi_1, \Xi_2 \vdash F_1^+ \otimes F_2^+ \ \mathsf{act}}$$

$$\frac{\Psi, \mathbf{S}_+(n, A) \vdash C}{\Psi, \mathbf{S}^+(n, A) \ \mathsf{act} \vdash C} \qquad \frac{}{\Gamma, \mathbf{S}_+(n, A) \vdash \mathbf{S}^+(n, A) \ \mathsf{act}}$$

$$\frac{\Gamma \vdash F^- \ \mathsf{res}}{\Gamma \vdash !F^- \ \mathsf{act}} \qquad \frac{\Psi, F^- \ \mathsf{tr} \vdash C}{\Psi, !F^- \ \mathsf{act} \vdash C}$$

# 4 Soundness

**Theorem 4.1 (Cut Admissibility)** *For any $\Xi_1, \Xi_2, \Xi, \Psi, J, C, F^{\pm}, A$, all of the following hold:*

*TL) If $\Xi_1 \vdash J$ and $\Xi_2, J \vdash C$, then $\Xi_1, \Xi_2 \vdash C$.*

*$P^+$) If $\Xi \vdash F^+ \ \mathsf{act}$ and $\Psi, F^+ \ \mathsf{act} \vdash C$, then $\Xi, \Psi \vdash C$.*

*$P^-$) If $\Psi \vdash F^- \ \mathsf{act}$ and $\Xi, F^- \ \mathsf{act} \vdash C$, then $\Xi, \Psi \vdash C$.*

*RC) If $\Xi \vdash J$ and $\Psi, J \vdash F \ \mathsf{act}$, then $\Xi, \Psi \vdash F \ \mathsf{act}$.*

*LCL*) *If* $\Xi \vdash J$ *and* $\Psi, J \vdash C$*, then* $\Xi, \Psi \vdash C$*.*

*LCR*) *If* $\Psi \vdash A$ *and* $\Xi, A \vdash C$*, then* $\Xi, \Psi \vdash C$*.*

**Proof** By lexicographic induction first on the cut judgment, and subsequently on the derivations involved. Order the derivations $\Xi, A \vdash C$ in the second input of $TL, LCR$ so that derivations whose last rule activates the cut formula $A$ are smaller than those that don't.

We write many cases in a pseudo-functional notation, writing the name of the case as a function symbol to indicate appeals to the induction hypothesis on a pair of derivations.

*TL*) **(Top-level Cut)** We analyze the possible cases. If either input derivation is the init rule, we are done:

$$\dfrac{\dfrac{}{\Gamma, p \vdash p} \qquad \begin{array}{c}\mathcal{D}'\\ \Gamma, \Delta_2, p \vdash C\end{array}}{\Gamma, \Delta_2, p \vdash C}\,\mathit{cut} \quad\longmapsto\quad \mathcal{D}'$$

$$\dfrac{\begin{array}{c}\mathcal{D}'\\ \Gamma, \Delta_1 \vdash p\end{array} \qquad \dfrac{}{\Gamma, p \vdash p}}{\Gamma, \Delta_1 \vdash p}\,\mathit{cut} \quad\longmapsto\quad \mathcal{D}'$$

We analyze cases first on the derivation of $\Xi_2, J \vdash C$. If it was an instance of the left activation rule, but not for the cut judgment, we do:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Xi_1 \vdash J\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{D}_2\\ \Xi_2, J, F^{\pm}\ \mathsf{act} \vdash C\end{array}}{\Xi_2, J, F^{\pm} \vdash C}}{\Xi_1, \Xi_2, F^{\pm} \vdash C}\,\mathit{cut} \quad\longmapsto\quad \dfrac{\begin{array}{c}LCL(\mathcal{D}_1, \mathcal{D}_2)\\ \Xi_1, \Xi_2, F^{\pm}\ \mathsf{act} \vdash C\end{array}}{\Xi_1, \Xi_2, F^{\pm} \vdash C}$$

If it was an instance of the copy rule, but not for the cut judgment, we do:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Xi_1, F^{-}\ \mathsf{tr} \vdash J\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{D}_2\\ \Xi_2, J, F^{-}\ \mathsf{tr}, F^{-}\ \mathsf{act} \vdash C\end{array}}{\Xi_2, J, F^{-}\ \mathsf{tr} \vdash C}}{\Xi_1, \Xi_2, F^{-}\ \mathsf{tr} \vdash C}\,\mathit{cut} \longmapsto \dfrac{\begin{array}{c}LCL(\mathcal{D}_1, \mathcal{D}_2)\\ \Xi_1, \Xi_2, F^{-}\ \mathsf{tr}, F^{-}\ \mathsf{act} \vdash C\end{array}}{\Xi_1, \Xi_2, F^{-}\ \mathsf{tr} \vdash C}$$

If it was an instance of the copy rule on cut judgment, we know that $\Xi_1 \vdash J$ is actually $\Gamma \vdash F^{-}\ \mathsf{res}$. The situation looks like

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash F^{-}\ \mathsf{res}\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{D}_2\\ \Xi, F^{-}\ \mathsf{tr}, F^{-}\ \mathsf{act} \vdash C\end{array}}{\Xi_2, F^{-}\ \mathsf{tr} \vdash C}}{\Gamma, \Xi_2 \vdash C}\,\mathit{cut}$$

We can invoke $LCL(\mathcal{D}_1, \mathcal{D}_2)$ to obtain $\Gamma, \Xi_2, F^- \ \mathsf{act} \vdash C$, and apply an inference rule to get a small derivation of $\Gamma, \Xi_2, F^- \ \mathsf{res} \vdash C$. The induction hypothesis $TL$ on $\mathcal{D}_1$ and this derivation yields $\Gamma, \Xi_2 \vdash C$ as required.

If it was an instance of the right activation rule, we do:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Xi_1 \vdash J\end{array} \quad \dfrac{\begin{array}{c}\mathcal{D}_2\\ \Xi_2, J \vdash F^{\pm} \ \mathsf{act}\end{array}}{\Xi_2, J \vdash F^{\pm}}}{\Xi_1, \Xi_2 \vdash C}\,cut \quad \longmapsto \quad \dfrac{\begin{array}{c}RC(\mathcal{D}_1, \mathcal{D}_2)\\ \Xi_1, \Xi_2, J \vdash F^{\pm} \ \mathsf{act}\end{array}}{\Xi_1, \Xi_2, J \vdash F^{\pm}}$$

The only remaining rule that could have ended the derivation of $\Xi_2, J \vdash C$ is an instance of the left activation rule for the cut judgment. Therefore we know $J$ is of the form $A \ \mathsf{res}$, not $F^- \ \mathsf{tr}$.

Consider possible cases for the derivation of $\Xi_1 \vdash A$. It could be an instance of the left activation rule, in which case we do:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Xi_1, F^{\pm} \ \mathsf{act} \vdash A\end{array}}{\Xi_1, F^{\pm} \vdash A} \quad \begin{array}{c}\mathcal{D}_2\\ \Xi_2, A \vdash C\end{array}}{\Xi_1, \Xi_2, F^{\pm} \vdash C}\,cut \quad \longmapsto \quad \dfrac{\begin{array}{c}LCR(\mathcal{D}_1, \mathcal{D}_2)\\ \Xi_1, \Xi_2, F^{\pm} \ \mathsf{act} \vdash C\end{array}}{\Xi_1, \Xi_2, F^{\pm} \vdash C}$$

Or it could be an instance of the copy rule, in which case we do:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Xi_1, F^- \ \mathsf{tr}, F^- \ \mathsf{act} \vdash A\end{array}}{\Xi, F^- \ \mathsf{tr} \vdash A} \quad \begin{array}{c}\mathcal{D}_2\\ \Xi_2, F^- \ \mathsf{tr}, A \vdash C\end{array}}{\Xi_1, \Xi_2, F^- \ \mathsf{tr} \vdash C}\,cut \quad \longmapsto$$

$$\dfrac{\begin{array}{c}LCR(\mathcal{D}_1, \mathcal{D}_2)\\ \Xi_1, \Xi_2, F^- \ \mathsf{tr}, F^- \ \mathsf{act} \vdash C\end{array}}{\Xi_1, \Xi_2, F^- \ \mathsf{tr} \vdash C}$$

The only remaining case is the 'principal cut':

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Xi_1 \vdash F^{\pm} \ \mathsf{act}\end{array}}{\Xi_1 \vdash F^{\pm}} \quad \dfrac{\begin{array}{c}\mathcal{D}_2\\ \Xi_2, F^{\pm} \ \mathsf{act} \vdash C\end{array}}{\Xi_2, F^{\pm} \vdash C}}{\Xi_1, \Xi_2 \vdash C}\,cut \quad \longmapsto \quad P^{\pm}(\mathcal{D}_1, \mathcal{D}_2)$$

$P^+$) **(Positive Principal Cut)** Based on the cut formula, we know exactly the last rule used on both input derivations. Here the appeal to the induction hypothesis is justified by the fact that the cut formula becomes smaller. The same comments apply to the negative principal cut cases below.

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \Xi_1 \vdash F_1^+\ \mathsf{act}\end{matrix}\quad \begin{matrix}\mathcal{D}_2\\ \Xi_2 \vdash F_2^+\ \mathsf{act}\end{matrix}}{\Xi_1, \Xi_2 \vdash F_1^+ \otimes F_2^+\ \mathsf{act}} \quad \dfrac{\begin{matrix}\mathcal{D}'\\ \Psi, F_1^+\ \mathsf{act}, F_2^+\ \mathsf{act} \vdash C\end{matrix}}{\Psi, F_1^+ \otimes F_2^+\ \mathsf{act} \vdash C}}{\Xi_1, \Xi_2, \Psi \vdash C}\,cut$$

$$\longmapsto \quad P^+(\mathcal{D}_1, P^+(\mathcal{D}_2, \mathcal{D}'))$$

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}'\\ \Xi \vdash F_i^+\ \mathsf{act}\end{matrix}}{\Xi \vdash F_1^+ \oplus F_2^+\ \mathsf{act}} \quad \dfrac{\begin{matrix}\mathcal{D}_1\\ \Psi, F_1^+\ \mathsf{act} \vdash C\end{matrix}\quad \begin{matrix}\mathcal{D}_2\\ \Psi, F_2^+\ \mathsf{act} \vdash C\end{matrix}}{\Psi, F_1^+ \oplus F_2^+\ \mathsf{act} \vdash C}}{\Xi, \Psi \vdash C}$$

$$\longmapsto \quad P^+(\mathcal{D}', \mathcal{D}_i)$$

$$\dfrac{\dfrac{}{\Gamma, \mathbf{S}_+(n, A) \vdash \mathbf{S}^+(n, A)\ \mathsf{act}} \quad \dfrac{\begin{matrix}\mathcal{D}'\\ \Psi, \mathbf{S}_+(n, A) \vdash C\end{matrix}}{\Gamma, \Psi, \mathbf{S}^+(n, A)\ \mathsf{act} \vdash C}}{\Gamma, \Psi, \mathbf{S}_+(n, A) \vdash C}\,cut \quad \longmapsto \quad \mathcal{D}'$$

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \Xi \vdash A\end{matrix}}{\Xi \vdash \mathbf{d}^+ A\ \mathsf{act}} \quad \dfrac{\begin{matrix}\mathcal{D}_2\\ \Psi, A \vdash C\end{matrix}}{\Psi, \mathbf{d}^+ A\ \mathsf{act} \vdash C}}{\Xi, \Psi \vdash C} \quad \longmapsto \quad LCL(\mathcal{D}_1, \mathcal{D}_2)$$

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \Gamma \vdash F^-\ \mathsf{res}\end{matrix}}{\Gamma \vdash !F^-\ \mathsf{act}} \quad \dfrac{\begin{matrix}\mathcal{D}_2\\ \Psi, F^-\ \mathsf{tr} \vdash C\end{matrix}}{\Psi, !F^-\ \mathsf{act} \vdash C}}{\Gamma, \Psi \vdash C} \quad \longmapsto \quad LCL(\mathcal{D}_1, \mathcal{D}_2)$$

$P^-$) **(Negative Principal Cut)**

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \Psi, F^+\ \mathsf{act} \vdash F^-\ \mathsf{act}\end{matrix}}{\Psi \vdash F^+ \multimap F^-\ \mathsf{act}} \quad \dfrac{\begin{matrix}\mathcal{D}_2\\ \Xi_2 \vdash F^+\ \mathsf{act}\end{matrix}\quad \begin{matrix}\mathcal{D}_3\\ \Xi_3, F^-\ \mathsf{act} \vdash C\end{matrix}}{\Xi_1, \Xi_2, F^+ \multimap F^-\ \mathsf{act} \vdash C}}{\Xi_1, \Xi_2, \Psi \vdash C}$$

$$\longmapsto \quad P^-(P^+(\mathcal{D}_2, \mathcal{D}_1), \mathcal{D}_3)$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Psi \vdash F_1^- \ \mathsf{act}\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \Psi \vdash F_2^- \ \mathsf{act}\end{array}}{\Psi \vdash F_1^- \mathbin{\&} F_2^- \ \mathsf{act}} \quad \dfrac{\begin{array}{c}\mathcal{D}'\\ \Xi, F_i^- \ \mathsf{act} \vdash C\end{array}}{\Xi, F_1^- \mathbin{\&} F_2^- \ \mathsf{act} \vdash C}}{\Xi, \Psi \vdash C}$$

$$\longmapsto \quad P^-(\mathcal{D}_i, \mathcal{D}')$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}'\\ \Psi \vdash \mathbf{S}_-(n, A)\end{array}}{\Psi \vdash \mathbf{S}^-(n, A) \ \mathsf{act}} \quad \dfrac{}{\Gamma, \mathbf{S}^-(n, A) \ \mathsf{act} \vdash \mathbf{S}_-(n, A)}}{\Gamma, \Psi \vdash \mathbf{S}_-(n, A)} \quad \longmapsto \quad \mathcal{D}'$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Psi \vdash A\end{array}}{\Psi \vdash \mathbf{d}^- A \ \mathsf{act}} \quad \dfrac{\begin{array}{c}\mathcal{D}_2\\ \Xi, A \vdash C\end{array}}{\Xi, \mathbf{d}^- A \ \mathsf{act} \vdash C}}{\Xi, \Psi \vdash C} \quad \longmapsto \quad LCR(\mathcal{D}_1, \mathcal{D}_2)$$

$RC$) **(Right Rule Commutative)**

If $\Xi \vdash J$ and $\Psi, J \vdash F$ $\mathsf{act}$, then $\Xi, \Psi \vdash F$ $\mathsf{act}$.

Note that $J$ is necessarily a *mobile* judgment.

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Xi \vdash J\end{array} \quad \dfrac{\begin{array}{c}\mathcal{D}_2\\ \Psi, J \vdash A\end{array}}{\Psi, J \vdash \mathbf{d}^{\pm} A \ \mathsf{act}}}{\Xi, \Psi \vdash \mathbf{d}^{\pm} A \ \mathsf{act}}\,\mathit{cut} \quad \longmapsto \quad \dfrac{\begin{array}{c}LCL(\mathcal{D}_1, \mathcal{D}_2)\\ \Xi, \Psi \vdash A\end{array}}{\Xi, \Psi \vdash \mathbf{d}^{\pm} A \ \mathsf{act}}$$

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Xi \vdash J\end{array} \quad \dfrac{\begin{array}{c}\mathcal{D}_2\\ \Psi, J, F^+ \ \mathsf{act} \vdash F^- \ \mathsf{act}\end{array}}{\Psi, J \vdash F^+ \multimap F^- \ \mathsf{act}}}{\Xi, \Psi \vdash F^+ \multimap F^- \ \mathsf{act}}\,\mathit{cut} \quad \longmapsto \quad \dfrac{\begin{array}{c}RC(\mathcal{D}_1, \mathcal{D}_2)\\ \Xi, \Psi, F^+ \vdash F^- \ \mathsf{act}\end{array}}{\Xi, \Psi \vdash F^+ \multimap F^- \ \mathsf{act}}$$

$$\dfrac{\begin{array}{c}\mathcal{D}'\\ \Xi \vdash J\end{array} \quad \dfrac{\begin{array}{c}\mathcal{D}_1\\ \Psi, J \vdash F_1^- \ \mathsf{act}\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \Psi, J \vdash F_2^- \ \mathsf{act}\end{array}}{\Psi, J \vdash F_1^- \mathbin{\&} F_2^- \ \mathsf{act}}}{\Xi, \Psi \vdash F_1^- \mathbin{\&} F_2^- \ \mathsf{act}}\,\mathit{cut} \mapsto \dfrac{\begin{array}{c}RC(\mathcal{D}', \mathcal{D}_1)\\ \Xi, \Psi \vdash F_1^- \ \mathsf{act}\end{array} \quad \begin{array}{c}RC(\mathcal{D}', \mathcal{D}_2)\\ \Xi, \Psi \vdash F_2^- \ \mathsf{act}\end{array}}{\Xi, \Psi \vdash F_1^- \mathbin{\&} F_2^- \ \mathsf{act}}$$

$$\dfrac{\dfrac{\mathcal{D}_1}{\Xi \vdash J} \quad \dfrac{\dfrac{\mathcal{D}_2}{\Psi, J \vdash \mathbf{S}_{-}(n, A)}}{\Psi, J \vdash \mathbf{S}^{-}(n, A)\ \mathsf{act}}}{\Xi, \Psi \vdash \mathbf{S}^{-}(n, A)\ \mathsf{act}}\ \mathit{cut} \quad \longmapsto \quad \dfrac{\dfrac{LCL(\mathcal{D}_1, \mathcal{D}_2)}{\Xi, \Psi \vdash \mathbf{S}_{-}(n, A)}}{\Xi, \Psi \vdash \mathbf{S}^{-}(n, A)\ \mathsf{act}}$$

There are three cases for $\otimes$, depending on whether $J$ propagates to the left, the right, or both branches.

Left:

$$\dfrac{\dfrac{\mathcal{D}'}{\Xi' \vdash J} \quad \dfrac{\dfrac{\mathcal{D}_1}{\Xi_1, J \vdash F_1^+} \quad \dfrac{\mathcal{D}_2}{\Xi_2 \vdash F_2^+}}{\Xi_1, \Xi_2, J \vdash F_1^+ \otimes F_2^+}}{\Xi', \Xi_1, \Xi_2 \vdash F_1^+ \otimes F_2^+}\ \mathit{cut} \quad \longmapsto \quad \dfrac{\dfrac{RC(\mathcal{D}', \mathcal{D}_1)}{\Xi', \Xi_1 \vdash F_1^+} \quad \dfrac{\mathcal{D}_2}{\Xi_2 \vdash F_2^+}}{\Xi', \Xi_1, \Xi_2 \vdash F_1^+ \otimes F_2^+}$$

Right:

$$\dfrac{\dfrac{\mathcal{D}'}{\Xi' \vdash J} \quad \dfrac{\dfrac{\mathcal{D}_1}{\Xi_1 \vdash F_1^+} \quad \dfrac{\mathcal{D}_2}{\Xi_2, J \vdash F_2^+}}{\Xi_1, \Xi_2, J \vdash F_1^+ \otimes F_2^+}}{\Xi', \Xi_1, \Xi_2 \vdash F_1^+ \otimes F_2^+}\ \mathit{cut} \quad \longmapsto \quad \dfrac{\dfrac{\mathcal{D}_1}{\Xi_1 \vdash F_1^+} \quad \dfrac{RC(\mathcal{D}', \mathcal{D}_2)}{\Xi', \Xi_2 \vdash F_2^+}}{\Xi', \Xi_1, \Xi_2 \vdash F_1^+ \otimes F_2^+}$$

Both:

$$\dfrac{\dfrac{\mathcal{D}'}{\Gamma \vdash G^-\ \mathsf{res}} \quad \dfrac{\dfrac{\mathcal{D}_1}{\Xi_1, G^-\ \mathsf{tr} \vdash F_1^+} \quad \dfrac{\mathcal{D}_2}{\Xi_2, G^-\ \mathsf{tr} \vdash F_2^+}}{\Xi_1, \Xi_2, G^-\ \mathsf{tr} \vdash F_1^+ \otimes F_2^+}}{\Gamma, \Xi_1, \Xi_2 \vdash F_1^+ \otimes F_2^+}\ \mathit{cut}$$

$$\longmapsto \quad \dfrac{\dfrac{RC(\mathcal{D}', \mathcal{D}_1)}{\Gamma, \Xi_1 \vdash F_1^+} \quad \dfrac{RC(\mathcal{D}', \mathcal{D}_2)}{\Gamma, \Xi_2 \vdash F_2^+}}{\Gamma, \Xi_1, \Xi_2 \vdash F_1^+ \otimes F_2^+}$$

$$\dfrac{\dfrac{\mathcal{D}'}{\Xi_1 \vdash J} \quad \dfrac{\dfrac{\mathcal{D}_1}{\Xi_2, J \vdash F_i^+\ \mathsf{act}}}{\Xi_2, J \vdash F_1^+ \oplus F_2^+\ \mathsf{act}}}{\Xi_1, \Xi_2 \vdash F_1^+ \oplus F_2^+\ \mathsf{act}}\ \mathit{cut} \quad \longmapsto \quad \dfrac{\dfrac{RC(\mathcal{D}_1, \mathcal{D}_2)}{\Xi_1, \Xi_2 \vdash F_i^+\ \mathsf{act}}}{\Xi_1, \Xi_2 \vdash F_1^+ \oplus F_2^+\ \mathsf{act}}$$

*(No case for* $\mathbf{S}^{+}(A)\ \mathsf{act}$*)*

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash G^- \ \mathsf{res}\end{array} \quad \dfrac{\begin{array}{c}\mathcal{D}_2\\ \Gamma, G^- \ \mathsf{tr} \vdash F^- \ \mathsf{res}\end{array}}{\Gamma, G^- \ \mathsf{tr} \vdash\ !F^- \ \mathsf{act}}}{\Gamma \vdash\ !F^- \ \mathsf{act}}\ cut \qquad \longmapsto \qquad \dfrac{\begin{array}{c} RC(\mathcal{D}_1, \mathcal{D}_2)\\ \Gamma \vdash\ !F^- \ \mathsf{res}\end{array}}{\Gamma \vdash\ !F^- \ \mathsf{act}}$$

$LCL$) **(Left Rule Commutative, Cut Formula on Left)** If $\Psi$ has no $\mathsf{act}$ assumptions, appeal to $TL$ on the same derivations. Otherwise split cases on the rightmost proposition in $\Psi$.

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Xi \vdash J\end{array} \quad \dfrac{\begin{array}{c}\mathcal{D}_2\\ \Psi, J, A \vdash C\end{array}}{\Psi, J, \mathbf{d}\, A \ \mathsf{act} \vdash C}}{\Xi, \Psi, \mathbf{d}\, A \ \mathsf{act} \vdash C}\ cut \qquad \longmapsto \qquad \dfrac{\begin{array}{c} LCL(\mathcal{D}_1, \mathcal{D}_2)\\ \Xi, \Psi, A \vdash C\end{array}}{\Xi, \Psi, \mathbf{d}\, A \ \mathsf{act} \vdash C}$$

For $\multimap$, we need to split on subcases depending on whether $J$ propagates to the left branch, or the right branch, or both.

Left:

$$\dfrac{\begin{array}{c}\mathcal{D}'\\ \Xi' \vdash A\end{array} \quad \dfrac{\begin{array}{c}\mathcal{D}_1\\ \Xi_1, A \vdash F^+ \ \mathsf{act}\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \Xi_2, F^- \ \mathsf{act} \vdash C\end{array}}{\Xi_1, \Xi_2, A, F^+ \multimap F^- \ \mathsf{act} \vdash C}}{\Xi', \Xi_1, \Xi_2, F^+ \multimap F^- \ \mathsf{act} \vdash C}\ cut$$

$$\longmapsto \quad \dfrac{\begin{array}{c}LCL(\mathcal{D}', \mathcal{D}_1)\\ \Xi', \Xi_1 \vdash F^+ \ \mathsf{act}\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \Xi_2, F^- \ \mathsf{act} \vdash C\end{array}}{\Xi', \Xi_1, \Xi_2, F^+ \multimap F^- \ \mathsf{act} \vdash C}$$

Right:

$$\dfrac{\begin{array}{c}\mathcal{D}'\\ \Xi' \vdash A\end{array} \quad \dfrac{\begin{array}{c}\mathcal{D}_1\\ \Xi_1 \vdash F^+ \ \mathsf{act}\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \Xi_2, A, F^- \ \mathsf{act} \vdash C\end{array}}{\Xi_1, \Xi_2, A, F^+ \multimap F^- \ \mathsf{act} \vdash C}}{\Xi', \Xi_1, \Xi_2, F^+ \multimap F^- \ \mathsf{act} \vdash C}\ cut$$

$$\longmapsto \quad \dfrac{\begin{array}{c}\mathcal{D}_1\\ \Xi_1 \vdash F^+ \ \mathsf{act}\end{array} \quad \begin{array}{c}LCL(\mathcal{D}', \mathcal{D}_2)\\ \Xi', \Xi_2, F^- \ \mathsf{act} \vdash C\end{array}}{\Xi', \Xi_1, \Xi_2, F^+ \multimap F^- \ \mathsf{act} \vdash C}$$

Both:

$$\dfrac{\begin{array}{c}\mathcal{D}'\\ \Gamma \vdash G^- \ \mathsf{res}\end{array} \quad \dfrac{\begin{array}{c}\mathcal{D}_1\\ \Xi_1, G^- \ \mathsf{tr} \vdash F^+ \ \mathsf{act}\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \Xi_2, G^- \ \mathsf{tr}, F^- \ \mathsf{act} \vdash C\end{array}}{\Xi_1, \Xi_2, G^- \ \mathsf{tr}, F^+ \multimap F^- \ \mathsf{act} \vdash C}}{\Gamma, \Xi_1, \Xi_2, F^+ \multimap F^- \ \mathsf{act} \vdash C}\ cut$$

$$\longmapsto \quad \dfrac{\stackrel{LCL(\mathcal{D}',\mathcal{D}_1)}{\Gamma,\Xi_1 \vdash F^+ \ \mathsf{act}} \qquad \stackrel{LCL(\mathcal{D}',\mathcal{D}_2)}{\Gamma,\Xi_2 F^- \ \mathsf{act} \vdash C}}{\Gamma,\Xi_1,\Xi_2,F^+ \multimap F^- \ \mathsf{act} \vdash C}$$

$$\dfrac{\stackrel{\mathcal{D}_1}{\Xi_1 \vdash J} \qquad \dfrac{\stackrel{\mathcal{D}_2}{\Xi_2, J, F_i^- \ \mathsf{act} \vdash C}}{\Xi_2, J, F_1^- \ \& \ F_2^- \ \mathsf{act} \vdash C}}{\Xi_1,\Xi_2,F_1^- \ \& \ F_2^- \ \mathsf{act} \vdash C}\,cut \quad \longmapsto \quad \dfrac{\stackrel{LCL(\mathcal{D}_1,\mathcal{D}_2)}{\Xi_1,\Xi_2,F_i^- \ \mathsf{act} \vdash C}}{\Xi_1,\Xi_2,F_1^- \ \& \ F_2^- \ \mathsf{act} \vdash C}$$

*(No case for* $\mathbf{S}^-(n,A)$ act*)*

$$\dfrac{\stackrel{\mathcal{D}'}{\Psi \vdash J} \qquad \dfrac{\stackrel{\mathcal{D}_1}{\Xi, J, F_1^+ \ \mathsf{act} \vdash C} \qquad \stackrel{\mathcal{D}_2}{\Xi, J, F_2^+ \ \mathsf{act} \vdash C}}{\Xi, J, F_1^+ \oplus F_2^+ \ \mathsf{act} \vdash C}}{\Psi,\Xi,F_1^+ \oplus F_2^+ \ \mathsf{act} \vdash C}\,cut$$

$$\longmapsto \quad \dfrac{\stackrel{LCL(\mathcal{D}',\mathcal{D}_1)}{\Psi,\Xi,F_1^+ \ \mathsf{act} \vdash C} \qquad \stackrel{LCL(\mathcal{D}',\mathcal{D}_2)}{\Psi,\Xi,F_2^+ \ \mathsf{act} \vdash C}}{\Psi,\Xi,F_2^+ \oplus F_2^+ \ \mathsf{act} \vdash C}$$

$$\dfrac{\stackrel{\mathcal{D}_1}{\Xi \vdash J} \qquad \dfrac{\stackrel{\mathcal{D}_2}{\Psi, J, F_1^+ \ \mathsf{act}, F_2^+ \ \mathsf{act} \vdash C}}{\Psi, J, F_1^+ \otimes F_2^+ \ \mathsf{act} \vdash C}}{\Xi,\Psi,F_1^+ \otimes F_2^+ \ \mathsf{act} \vdash C}\,cut$$

$$\longmapsto \quad \dfrac{\stackrel{LCL(\mathcal{D}_1,\mathcal{D}_2)}{\Xi,\Psi,F_1^+ \ \mathsf{act}, F_2^+ \ \mathsf{act} \vdash C}}{\Xi,\Psi,F_1^+ \otimes F_2^+ \ \mathsf{act} \vdash C}\,cut$$

$$\dfrac{\stackrel{\mathcal{D}_1}{\Xi \vdash J} \qquad \dfrac{\stackrel{\mathcal{D}_2}{\Psi, J, \mathbf{S}_+(n,A) \vdash C}}{\Psi, J, \mathbf{S}^+(n,A) \ \mathsf{act} \vdash C}}{\Xi,\Psi,\mathbf{S}^+(n,A) \ \mathsf{act} \vdash C}\,cut \quad \longmapsto \quad \dfrac{\stackrel{LCL(\mathcal{D}_1,\mathcal{D}_2)}{\Xi,\Psi,\mathbf{S}_+(n,A) \vdash C}}{\Xi,\Psi,\mathbf{S}^+(n,A) \ \mathsf{act} \vdash C}\,cut$$

$$\dfrac{\stackrel{\mathcal{D}_1}{\Xi \vdash J} \qquad \dfrac{\stackrel{\mathcal{D}_2}{\Psi, J, F^- \ \mathsf{tr} \vdash C}}{\Psi, J, !F^- \ \mathsf{act} \vdash C}}{\Xi,\Psi !F^- \ \mathsf{act} \vdash C}\,cut \quad \longmapsto \quad \dfrac{\stackrel{LCL(\mathcal{D}_1,\mathcal{D}_2)}{\Xi,\Psi,F^- \ \mathsf{tr} \vdash C}}{\Xi,\Psi,!F^- \ \mathsf{act} \vdash C}$$

*LCR*) **(Left Rule Commutative, Cut Formula on Right)** If $\Psi$ has no $\mathsf{act}$ assumptions, appeal to $TL$ on the same derivations. Otherwise split cases on the rightmost proposition in $\Psi$.

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\Psi, B \vdash A}}{\Psi, \mathbf{d}^{\pm} B\ \mathsf{act} \vdash A} \quad \overset{\mathcal{D}_2}{\Xi, A \vdash C}}{\Xi, \Psi, \mathbf{d}^{\pm} B\ \mathsf{act} \vdash C}\,cut \quad \longmapsto \quad \dfrac{\overset{LCR(\mathcal{D}_1, \mathcal{D}_2)}{\Xi, \Psi, B \vdash C}}{\Xi, \Psi, \mathbf{d}^{\pm} B\ \mathsf{act} \vdash C}$$

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\Xi_1 \vdash F^+\ \mathsf{act}} \quad \overset{\mathcal{D}_2}{\Xi_2, F^-\ \mathsf{act} \vdash A}}{\Xi_1, \Xi_2, F^+ \multimap F^-\ \mathsf{act} \vdash A} \quad \overset{\mathcal{D}'}{\Xi', A \vdash C}}{\Xi_1, \Xi_2, \Xi', F^+ \multimap F^-\ \mathsf{act} \vdash A}\,cut$$

$$\longmapsto \quad \dfrac{\overset{\mathcal{D}_1}{\Xi_1 \vdash F^+\ \mathsf{act}} \quad \overset{LCR(\mathcal{D}_2, \mathcal{D}')}{\Xi_2, \Xi', F^-\ \mathsf{act} \vdash A}}{\Xi_1, \Xi_2, \Xi', F^+ \multimap F^-\ \mathsf{act} \vdash A}$$

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\Xi_1, F_i^-\ \mathsf{act} \vdash A}}{\Xi_1, F_1^- \ \&\ F_2^-\ \mathsf{act} \vdash A} \quad \overset{\mathcal{D}_2}{\Xi, A \vdash C}}{\Xi_1, \Xi_2, F_1^- \ \&\ F_2^-\ \mathsf{act} \vdash C}\,cut \quad \longmapsto \quad \dfrac{\overset{LCR(\mathcal{D}_1, \mathcal{D}_2)}{\Xi_1, \Xi_2, F_i^-\ \mathsf{act} \vdash C}}{\Xi_1, \Xi_2, F_1^- \ \&\ F_2^-\ \mathsf{act} \vdash C}$$

*(No case for* $\mathbf{S}^-(n, A)\ \mathsf{act}$*)*

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\Psi, F_1^+\ \mathsf{act} \vdash A} \quad \overset{\mathcal{D}_2}{\Psi, F_2^+\ \mathsf{act} \vdash A}}{\Psi, F_1^+ \oplus F_2^+\ \mathsf{act} \vdash A} \quad \overset{\mathcal{D}'}{\Xi, A \vdash C}}{\Xi, \Psi, F_1^+ \oplus F_2^+\ \mathsf{act} \vdash C}\,cut$$

$$\longmapsto \quad \dfrac{\overset{LCR(\mathcal{D}_1, \mathcal{D}')}{\Xi, \Psi, F_1^+\ \mathsf{act} \vdash C} \quad \overset{LCR(\mathcal{D}_2, \mathcal{D}')}{\Xi, \Psi, F_2^+\ \mathsf{act} \vdash C}}{\Xi, \Psi, F_1^+ \oplus F_2^+\ \mathsf{act} \vdash C}$$

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\Psi, F_1^+\ \mathsf{act}, F_2^+\ \mathsf{act} \vdash A}}{\Psi, F_1^+ \otimes F_2^+ \vdash A} \quad \overset{\mathcal{D}_2}{\Xi, A \vdash C}}{\Xi, \Psi, F_1^+ \otimes F_2^+\ \mathsf{act} \vdash C}\,cut$$

$$\longmapsto \quad \dfrac{\begin{array}{c}LCR(\mathcal{D}_1, \mathcal{D}_2)\\ \Xi, \Psi, F_1^+\ \mathsf{act}, F_2^+\ \mathsf{act} \vdash C\end{array}}{\Xi, \Psi, F_1^+ \otimes F_2^+\ \mathsf{act} \vdash C}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Psi, \mathbf{S}_+(n, A) \vdash A\end{array}}{\Psi, \mathbf{S}^+(n, A)\ \mathsf{act} \vdash A} \quad \begin{array}{c}\mathcal{D}_2\\ \Xi, A \vdash C\end{array}}{\Xi, \Psi, \mathbf{S}^+(n, A)\ \mathsf{act} \vdash C}\ cut \quad \longmapsto \quad \dfrac{\begin{array}{c}LCR(\mathcal{D}_1, \mathcal{D}_2)\\ \Xi, \Psi, \mathbf{S}_+(n, A) \vdash C\end{array}}{\Xi, \Psi, \mathbf{S}^+(n, A)\ \mathsf{act} \vdash C}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Psi, F^-\ \mathsf{tr} \vdash A\end{array}}{\Psi, !F^-\ \mathsf{act} \vdash A} \quad \begin{array}{c}\mathcal{D}_2\\ \Xi, A \vdash C\end{array}}{\Xi, \Psi, !F^-\ \mathsf{act} \vdash C}\ cut \quad \longmapsto \quad \dfrac{\begin{array}{c}LCR(\mathcal{D}_1, \mathcal{D}_2)\\ \Xi, \Psi, F^-\ \mathsf{tr} \vdash C\end{array}}{\Xi, \Psi, !F^-\ \mathsf{act} \vdash C}$$

■

## 5 Completeness

Here are a few expedient refinments of the existing syntactic categories:

$$\begin{array}{rcl}\text{Right sides } \gamma & ::= & A\ \mathsf{res} \mid F^-\ \mathsf{act}\\ \text{Passives } a_\pm & ::= & p \mid \mathbf{S}_\pm(n, A)\\ \text{Passive contexts } \xi & ::= & \cdot \mid \xi, a_+\end{array}$$

Here by 'passive' it is meant things that can not be activated.

In order to show completeness, we define several relations

$$\begin{array}{llll}A \rightharpoonup B & F^\pm \rightharpoonup^\pm G^\pm & \Psi \rightharpoonup^{\mathsf{ctx}} \Xi & \gamma \rightharpoonup^{\mathsf{conc}} C\\ & A \rightharpoonup_\pm a_\pm & \Psi \rightharpoonup_{\mathsf{ctx}} \xi & A \rightharpoonup_{\mathsf{conc}} a_-\end{array}$$

which in some sense 'erase' the choice of how propostions were coalesced together. The $\rightharpoonup$-relations cooperate to insert a lot of $\mathbf{d}^\pm$s to accomplish this. Completeness works by 'pulling back' provability of things on the right of $\rightharpoonup$s to the left. The difference between the subscript and superscript versions of the relations corresponds to different inductive stages of the completeness proof. The subscript versions are stronger, and allow only expressions that are already more 'passive'.

They are defined by

$$\dfrac{F^\pm \rightharpoonup^\pm G^\pm}{F^\pm \rightharpoonup G^\pm} \qquad \dfrac{}{p \rightharpoonup p}$$

$$\dfrac{A \rightharpoonup B}{A \rightharpoonup_\pm \mathbf{S}_\pm(0, B)} \qquad \dfrac{}{\mathbf{S}_\pm(n, p) \rightharpoonup_\pm \mathbf{S}_\pm(n+1, p)}$$

$$\frac{F^- \rightharpoonup^- G^-}{F^- \ \mathsf{act} \rightharpoonup^{\mathsf{conc}} G^-} \qquad \frac{A \rightharpoonup_{\mathsf{conc}} C}{A \rightharpoonup^{\mathsf{conc}} C} \qquad \frac{A \rightharpoonup_- C}{A \rightharpoonup_{\mathsf{conc}} C}$$

$$\frac{\Psi \rightharpoonup_{\mathsf{ctx}} \Xi \qquad A \rightharpoonup_+ B}{(\Psi, A) \rightharpoonup_{\mathsf{ctx}} (\Xi, B)} \qquad \frac{}{() \rightharpoonup_{\mathsf{ctx}} ()}$$

$$\frac{\Psi \rightharpoonup_{\mathsf{ctx}} \Xi \qquad F^- \rightharpoonup^- G^-}{(\Psi, F^- \ \mathsf{tr}) \rightharpoonup_{\mathsf{ctx}} (\Xi, \mathbf{S}^-(0, G^-) \ \mathsf{tr})}$$

$$\frac{\Psi \rightharpoonup^{\mathsf{ctx}} \Xi \qquad F^+ \rightharpoonup^+ G^+}{(\Psi, F^+ \ \mathsf{act}) \rightharpoonup^{\mathsf{ctx}} (\Xi, G^+)} \qquad \frac{\Psi \rightharpoonup_{\mathsf{ctx}} \Xi}{\Psi \rightharpoonup^{\mathsf{ctx}} \Xi}$$

$$\frac{}{\mathbf{S}^\pm(n, p) \rightharpoonup^\pm \mathbf{S}^\pm(n+1, p)}$$

$$\frac{F_1^+ \rightharpoonup^+ G_1^+ \qquad F_2^+ \rightharpoonup^+ G_2^+}{F_1^+ \otimes F_2^+ \rightharpoonup^+ \mathbf{d}^+ \, G_1^+ \otimes \mathbf{d}^+ \, G_2^+}$$

$$\frac{F_1^+ \rightharpoonup^+ G_1^+ \qquad F_2^+ \rightharpoonup^+ G_2^+}{F_1^+ \oplus F_2^+ \rightharpoonup^+ \mathbf{d}^+ \, G_1^+ \oplus \mathbf{d}^+ \, G_2^+}$$

$$\frac{F_1^- \rightharpoonup^- G_1^- \qquad F_2^- \rightharpoonup^- G_2^-}{F_1^- \mathbin{\&} F_2^- \rightharpoonup^- \mathbf{d}^- \, G_1^- \mathbin{\&} \mathbf{d}^- \, G_2^-}$$

$$\frac{F_1^+ \rightharpoonup^+ G_1^+ \qquad F_2^- \rightharpoonup^- G_2^-}{F_1^+ \multimap F_2^- \rightharpoonup^- \mathbf{d}^+ \, G_1^+ \multimap \mathbf{d}^- \, G_2^-}$$

$$\frac{F^\pm \rightharpoonup^\pm G^\pm}{\mathbf{d}^\pm \, F^\pm \rightharpoonup^\pm \mathbf{d}^\pm \, G^\pm}$$

$$\frac{A \rightharpoonup B}{\mathbf{d}^\pm \, A \rightharpoonup^\pm \mathbf{S}^\pm(0, B)}$$

$$\frac{F^- \rightharpoonup^- G^-}{!F^- \rightharpoonup^+ !(\mathbf{S}^-(0, G^-))}$$

**Lemma 5.1 (Simple Identity)**

1. *If* $A \rightharpoonup B$*, then* $B \vdash B$.

**Proof** Follows the usual identity theorem for unfocused logic, by induction on $B$. This is because every proposition in the image of $\rightharpoonup$ has deactivations $\mathbf{d}^\pm$ around every propositional connective, so that it essentially admits every unfocused proof. ■

**Lemma 5.2 (Completeness)**

1. *If* $F^+ \rightharpoonup^+ G^+$, $\Xi \vdash G^+ \ \mathsf{act}$, *and* $\Psi \rightharpoonup_{\mathsf{ctx}} \Xi$, *then* $\Psi \vdash F^+ \ \mathsf{act}$.
2. *Suppose* $F^+ \rightharpoonup^+ G^+$ *and* $\Xi \vdash G^+$. *If any of the following hold*
   (a) *If* $\Psi \rightharpoonup_{\mathsf{ctx}} \Xi$
   (b) *If* $\Psi \rightharpoonup^{\mathsf{ctx}} \Xi$

   *then* $\Psi \vdash F^+$.
3. *If* $F^- \rightharpoonup^- G^-$, $\Xi, G^- \ \mathsf{act} \vdash C$, $\Psi \rightharpoonup_{\mathsf{ctx}} \Xi$, $\gamma \rightharpoonup_{\mathsf{conc}} C$, *then* $\Psi, F^- \ \mathsf{act} \vdash \gamma$.
4. *Suppose* $F^- \rightharpoonup^- G^-$ *and* $\Xi, G^- \vdash C$. *If any of the following hold*
   (a) *If* $\Psi \rightharpoonup_{\mathsf{ctx}} \Xi$ *and* $\gamma \rightharpoonup_{\mathsf{conc}} C$
   (b) *If* $\Psi \rightharpoonup^{\mathsf{ctx}} \Xi$ *and* $\gamma \rightharpoonup_{\mathsf{conc}} C$
   (c) *If* $\Psi \rightharpoonup^{\mathsf{ctx}} \Xi$ *and* $\gamma \rightharpoonup^{\mathsf{conc}} C$

   *then* $\Psi, F^- \vdash \gamma$.
5. *If* $A_1 \rightharpoonup B_1$ *and* $A_2 \rightharpoonup B_2$, *and* $B_1 \vdash B_2$, *then* $A_1 \vdash A_2$.

**Proof** By lexicographic induction on the case, and the derivations involved.

1. The main split is on cases of the derivation $\mathcal{D}_{\rightharpoonup}$ of $F^+ \rightharpoonup^+ G^+$, naming $\mathcal{D}$ the derivation of $\Xi \vdash G^+ \ \mathsf{act}$. After we get past the cases of $\rightharpoonup^+$ that result in a $\mathbf{S}^+$, however, everything is more syntax-directed, and the final rules both $\mathcal{D}_{\rightharpoonup}$ and $\mathcal{D}$ are determined by the top-level connective of $G^+$, so we just describe the case analysis in terms of $\mathcal{D}$.

   Case:

   $$\mathcal{D}_{\rightharpoonup} = \frac{}{\mathbf{S}^+(n,p) \rightharpoonup^+ \mathbf{S}^+(n+1,p)}$$

   therefore it must be that

   $$\mathcal{D} = \frac{}{\mathbf{S}_+(n+1,p) \vdash \mathbf{S}^+(n+1,p) \ \mathsf{act}}$$

   Hence $\Psi$ must be $\mathbf{S}_+(n,p)$, and thus the goal is to show $\mathbf{S}_+(n,p) \vdash \mathbf{S}^+(n,p)$, but this is immediately derivable with a single rule application.

   Case:

   $$\mathcal{D}_{\rightharpoonup} = \frac{\begin{matrix}\mathcal{D}'_{\rightharpoonup} \\ A_2 \rightharpoonup B\end{matrix}}{\mathbf{d}^+ A_2 \rightharpoonup^+ \mathbf{S}^+(0,B)}$$

   and

   $$\mathcal{D} = \frac{}{\mathbf{S}_+(0,B) \vdash \mathbf{S}^+(0,B) \ \mathsf{act}}$$

   By inversion on the rules defining $\rightharpoonup_{\mathsf{ctx}}$ and $\rightharpoonup_+$, we know $\Psi$ must be a single hypothesis $A_1$ such that $A_1 \rightharpoonup B$. By induction hypothesis

(5) on this fact together with $\mathcal{D}'_{\rightharpoonup}$ and Lemma 5.1 we can conclude that there is a derivation $\mathcal{D}'' :: A_1 \vdash A_2$ and form the derivation

$$\dfrac{\begin{matrix}\mathcal{D}'' \\ A_1 \vdash A_2\end{matrix}}{A_1 \vdash \mathbf{d}^+ \, A_2 \ \mathsf{act}}$$

All the subsequent cases are determined by the top-level connective of $G^+$.

Case:

$$\mathcal{D} = \dfrac{\begin{matrix}\mathcal{D}' \\ \Xi \vdash G^+\end{matrix}}{\Xi \vdash \mathbf{d}^+ \, G^+ \ \mathsf{act}}$$

We know that $F^+$ is such that $F^+ \rightharpoonup^+ G^+$, and we must show $\Psi \vdash \mathbf{d}^+ \, F^+ \ \mathsf{act}$. Apply the induction hypothesis part (2a) to $\mathcal{D}'$ to obtain $\Psi \vdash F^+$ and construct the derivation

$$\dfrac{\Psi \vdash F^+}{\Psi \vdash \mathbf{d}^+ \, F^+ \ \mathsf{act}}$$

Case:

$$\mathcal{D} = \dfrac{\dfrac{\begin{matrix}\mathcal{D}_1 \\ \Xi_1 \vdash G_1^+\end{matrix}}{\Xi_1 \vdash \mathbf{d}^+ \, G_1^+ \ \mathsf{act}} \quad \dfrac{\begin{matrix}\mathcal{D}_2 \\ \Xi_2 \vdash G_2^+\end{matrix}}{\Xi_2 \vdash \mathbf{d}^+ \, G_2^+ \ \mathsf{act}}}{\Xi_1, \Xi_2 \vdash \mathbf{d}^+ \, G_1^+ \otimes \mathbf{d}^+ \, G_2^+ \ \mathsf{act}}$$

We know $F_i^+ \rightharpoonup^+ G_i^+$. Split $\Psi$ into $\Psi_1, \Psi_2$ such that $\Psi_i \rightharpoonup^{\mathsf{ctx}} \Xi_i$. Apply the induction hypothesis part (2a) to $\mathcal{D}_1, \mathcal{D}_2$ to obtain $\Psi_i \vdash F_i^+$. Construct the derivation

$$\dfrac{\dfrac{\Psi_1 \vdash F_1^+}{\Psi_1 \vdash F_1^+ \ \mathsf{act}} \quad \dfrac{\Psi_2 \vdash F_2^+}{\Psi_2 \vdash F_2^+ \ \mathsf{act}}}{\Psi_1, \Psi_2 \vdash F_1^+ \otimes F_2^+ \ \mathsf{act}}$$

Case:

$$\mathcal{D} = \dfrac{\dfrac{\begin{matrix}\mathcal{D}' \\ \Xi \vdash G_i^+\end{matrix}}{\Xi \vdash \mathbf{d}^+ \, G_i^+ \ \mathsf{act}}}{\Xi \vdash \mathbf{d}^+ \, G_1^+ \oplus \mathbf{d}^+ \, G_2^+ \ \mathsf{act}}$$

We know $F_i^+ \rightharpoonup^+ G_i^+$. Apply the induction hypothesis part (2a) to

$\mathcal{D}'$ to obtain $\Psi_i \vdash F_i^+$. Construct the derivation

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}'\\ \Psi \vdash F_i^+\end{array}}{\Psi \vdash F_i^+ \ \mathsf{act}}}{\Psi \vdash F_1^+ \oplus F_2^+ \ \mathsf{act}}$$

Case:

$$\mathcal{D} = \dfrac{\dfrac{\Gamma \vdash \mathbf{S}_-(0, G^-)}{\Gamma \vdash \mathbf{S}^-(0, G^-) \ \mathsf{act}}}{\Gamma \vdash !\mathbf{S}^-(0, G^-) \ \mathsf{act}}$$

In this case, for some $F_2^-$,

$$\mathcal{D}_\rightharpoonup = \dfrac{F_2^- \rightharpoonup^- G^-}{!F_2^- \rightharpoonup^+ !(\mathbf{S}^-(0, G^-))}$$

Further analyzing the remaining structure of the derivation $\mathcal{D}'$, the only rule that can be applied is a copy from $\Gamma$. Since $\Gamma$ arose from $\rightharpoonup_{\mathsf{ctx}}$, it only contains propositions of the form $\mathbf{S}^-(0, G'^-)$ for various $G'^-$ such that $F' \rightharpoonup G'$ and $F' \ \mathsf{tr} \in \Psi$. Since copied assumptions retain activation, this only succeeds if we copy $\mathbf{S}^-(0, G^-)$. Hence $F_1^- \ \mathsf{tr}$ must have been in $\Psi$, for some $F_1^-$ such that $F_1^- \rightharpoonup^- G^-$. Moreover $\Psi$ must consist only of $\mathsf{tr}$ assumptions. By Lemma 5.1, $G^- \vdash G^-$. By the induction hypothesis part(5), $F_1^- \vdash F_2^-$. By an evident weakening lemma, and the fact that $\Psi$ happens to have only true hypotheses, we have $\mathcal{D}'' :: \Psi \vdash F_2^- \ \mathsf{act}$. Therefore construct the derivation

$$\dfrac{\begin{array}{c}\mathcal{D}''\\ \Psi \vdash F_2^-\end{array}}{\Psi \vdash !F_2^- \ \mathsf{act}}$$

2.

(a) By inversion on the rules defining $\rightharpoonup_{\mathsf{ctx}}$, the only applicable rule is right activation. Appeal to the induction hypothesis part (1).

(b) Here we proceed by analyzing the top connective of the rightmost active assumption in $\Psi$. If there are none, we can appeal to the i.h. part (2a).

Case: $\Psi = \Psi', F_1^+ \otimes F_2^+ \ \mathsf{act}$. Hence $\Xi = \Xi', \mathbf{d}^+ G_1^+ \otimes \mathbf{d}^+ G_2^+$. It is easy to construct a derivation of $G_1^+, G_2^+ \vdash \mathbf{d}^+ G_1^+ \otimes \mathbf{d}^+ G_2^+$ by using Lemma 5.1, so by cut we have $\Xi', G_1^+, G_2^+ \vdash G^+$. By induction hypothesis (2b) we obtain $\Psi', F_1^+ \ \mathsf{act}, F_2^+ \ \mathsf{act} \vdash F^+$. By rule, $\Psi', F_1^+ \otimes F_2^+ \ \mathsf{act} \vdash F^+$.

Case: $\Psi = \Psi', F_1^+ \oplus F_2^+\ \mathsf{act}$. Hence $\Xi = \Xi', \mathbf{d}^+ G_1^+ \oplus \mathbf{d}^+ G_2^+$. It is easy to construct a derivation of $G_i^+ \vdash \mathbf{d}^+ G_1^+ \oplus \mathbf{d}^+ G_2^+$ for both $i$ using Lemma 5.1, so by cut we have $\Xi', G_i^+ \vdash G^+$ for both $i$. By induction hypothesis (2b) we obtain $\Psi', F_i^+\ \mathsf{act} \vdash F^+$. By rule, $\Psi', F_1^+ \oplus F_2^+\ \mathsf{act} \vdash F^+$.

Case: $\Psi = \Psi', !F_0^-\ \mathsf{act}$. Hence $\Xi = \Xi', !(\mathbf{S}^-(0, G_0^-))$. It is easy to construct a derivation of $\mathbf{S}^-(0, G_0^-)\ \mathsf{tr} \vdash !(\mathbf{S}(0, G_0^-))$ by using Lemma 5.1, so by cut we have $\Xi', \mathbf{S}^-(0, G_0^-)\ \mathsf{tr} \vdash G^+$. By induction hypothesis (2b) we obtain $\Psi', F_0^-\ \mathsf{tr} \vdash F^+$. By rule, $\Psi', !(F_0^-)\ \mathsf{act} \vdash F^+$.

Case: $\Psi = \Psi', \mathbf{d}^+ A\ \mathsf{act}$. Here there are two subcases depending on the derivation $\Psi \rightharpoonup^{\mathsf{ctx}} \Xi$.

In the first, $\Xi = \Xi', \mathbf{d}^+ G_0^+$, and $A$ is of the form $F_0^+$ such that $F_0^+ \rightharpoonup^+ G_0^+$ for some $G_0^+$. It is easy to construct a derivation of $G_0^+ \vdash \mathbf{d}^+ G_0^+$ by using Lemma 5.1, so by cut we have $\Xi', G_0^+ \vdash G^+$. By induction hypothesis (2b) we obtain $\Psi', F_0^+\ \mathsf{act} \vdash F^+$. By one rule application, $\Psi', F_0^+ \vdash F^+$. By another, $\Psi', \mathbf{d}^+ F_0^+\ \mathsf{act} \vdash F^+$.

In the second, $\Xi = \Xi', \mathbf{S}^+(0, B)$ and $A \rightharpoonup B$ for some $A, B$. It is trivial to construct a derivation of $\mathbf{S}_+(0, B) \vdash \mathbf{S}^+(0, B)$ using Lemma 5.1 so by cut we have $\Xi', \mathbf{S}_+(0, B) \vdash G^+$. By induction hypothesis (2b) we obtain $\Psi', B \vdash F^+$. By rule, $\Psi', \mathbf{d}^+ B\ \mathsf{act} \vdash F^+$.

Case: $\Psi = \Psi', \mathbf{S}^+(n, A)\ \mathsf{act}$. We know $\Xi = \Xi', \mathbf{S}^+(n+1, B)$. It is trivial to construct a derivation of $\mathbf{S}_+(n+1, B) \vdash \mathbf{S}^+(n+1, B)$ using Lemma 5.1 so by cut we have $\Xi', \mathbf{S}_+(n+1, B) \vdash G^+$. By induction hypothesis (2b) we obtain $\Psi', \mathbf{S}^+(n, B)\ \mathsf{act} \vdash F^+$.

3. By the definition of $\rightharpoonup_{\mathsf{ctx}}$, the only propositions in $\Xi$ are of the form $\mathbf{S}^-(0, A)\ \mathsf{tr}$, $\mathbf{S}_+(0, A)$ and $\mathbf{S}_+(n, p)$ for $n \geq 1$. Split cases on the derivation $\mathcal{D}$ of $\Xi, G^-\ \mathsf{act} \vdash C$, and subsequently on $F^- \rightharpoonup^- G^-$ if necessary.

Case: $\mathcal{D}$ is a copy of some assumption $\mathbf{S}^-(0, G_0^-)\ \mathsf{tr}$. Then $C = \mathbf{S}_-(0, G_0^-)$, and $\Xi$ is empty of any resource assumptions. $\Psi$ is therefore of the form $\Gamma, F_1^-\ \mathsf{tr}$ such that $F_1^- \rightharpoonup^- G_0^-$. The $\gamma$ such that $\gamma \rightharpoonup_- C$ must be of the form $F_2^-$ where $F_2^- \rightharpoonup^- G_0^-$. We must show $\Gamma, F_1^-\ \mathsf{tr} \vdash F_2^-$, but this follows from the induction hypothesis (5), and weakening.

Case: The derivation of $F^- \rightharpoonup^- G^-$ is

$$\mathbf{S}^-(n, p) \rightharpoonup^- \mathbf{S}^-(n-1, p)$$

and

$$\mathcal{D} = \frac{}{\mathbf{S}^-(n-1, p)\ \mathsf{act} \vdash \mathbf{S}_-(n-1, p)}$$

Hence $\Psi$ must be $\mathbf{S}_-(n, p)$, and the goal is to show $\mathbf{S}_-(n, p) \vdash \mathbf{S}^-(n, p)$, which follows immediately.

Case: The derivation of $F^- \rightharpoonup^- G^-$ is

$$\dfrac{\begin{array}{c}\mathcal{D}' \\ A_1 \rightharpoonup B\end{array}}{\mathbf{d}^- \, A_1 \rightharpoonup^- \mathbf{S}^-(0, B)}$$

and

$$\mathcal{D} = \dfrac{}{\mathbf{S}^-(0, B) \ \mathsf{act} \vdash \mathbf{S}_-(0, B)}$$

By inversion on the rules defining $\rightharpoonup_{\mathsf{conc}}$, we know $\gamma$ must be $A_2$ such that $A_2 \rightharpoonup B$. By induction hypothesis (5) on this fact together with $\mathcal{D}'$ and Lemma 5.1 we can conclude that there is a derivation $\mathcal{D}'' :: A_1 \vdash A_2$ and form the derivation

$$\dfrac{\begin{array}{c}\mathcal{D}'' \\ A_1 \vdash A_2\end{array}}{\mathbf{d}^- \, A_1 \ \mathsf{act} \vdash A_2}$$

Case:

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}' \\ \Xi, G^- \vdash C\end{array}}{\Xi, \mathbf{d}^- \, G^- \ \mathsf{act} \vdash C}$$

We know that $F^-$ is such that $F^- \rightharpoonup^- G^-$, and we must show $\Psi, \mathbf{d}^- \, F^- \ \mathsf{act} \vdash \gamma$. Apply the induction hypothesis part (4c) to $\mathcal{D}'$ to obtain $\Psi, F^- \vdash \gamma$ and construct the derivation

$$\dfrac{\Psi, F^- \vdash \gamma}{\Psi, \mathbf{d}^- \, F^- \ \mathsf{act} \vdash \gamma}$$

Case:

$$\mathcal{D} = \dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1 \\ \Xi, G_1^- \vdash C\end{array}}{\Xi, \mathbf{d}^- \, G_1^- \ \mathsf{act} \vdash C} \quad \dfrac{\begin{array}{c}\mathcal{D}_2 \\ \Xi, G_2^- \vdash C\end{array}}{\Xi, \mathbf{d}^- \, G_2^- \ \mathsf{act} \vdash C}}{\Xi, \mathbf{d}^- \, G_1^- \ \& \ \mathbf{d}^- \, G_2^- \ \mathsf{act} \vdash C}$$

We know $F_i^- \rightharpoonup^- G_i^-$. Apply the induction hypothesis part (4c) to $\mathcal{D}_1, \mathcal{D}_2$ to obtain $\Psi_i, F_i^- \vdash \gamma$. Construct the derivation

$$\dfrac{\dfrac{\Psi, F_1^- \vdash \gamma}{\Psi, F_1^- \ \mathsf{act} \vdash \gamma} \quad \dfrac{\Psi, F_2^- \vdash \gamma}{\Psi, F_2^- \ \mathsf{act} \vdash \gamma}}{\Psi, F_1^- \ \& \ F_2^- \ \mathsf{act} \vdash \gamma}$$

Case:

$$\mathcal{D} = \cfrac{\cfrac{\begin{matrix}\mathcal{D}_1\\ \Xi_1 \vdash G^+\end{matrix}}{\Xi_1 \vdash \mathbf{d}^+ G^+ \ \mathsf{act}} \quad \cfrac{\begin{matrix}\mathcal{D}_2\\ \Xi_2, G^- \vdash C\end{matrix}}{\Xi_2, \mathbf{d}^- G^- \ \mathsf{act} \vdash C}}{\Xi_1, \Xi_2, \mathbf{d}^- G^+ \multimap \mathbf{d}^- G^- \ \mathsf{act} \vdash C}$$

We know $F^{\pm} \rightharpoonup^{\pm} G^{\pm}$. Apply the induction hypothesis part (2b) and (4c) to $\mathcal{D}_1, \mathcal{D}_2$ respectively to obtain $\Psi_1 \vdash F^+$ and $\Psi_2, F^- \vdash \gamma$. Construct the derivation

$$\cfrac{\cfrac{\Psi_1 \vdash F^+}{\Psi_1 \vdash F^+ \ \mathsf{act}} \quad \cfrac{\Psi_2, F^- \vdash \gamma}{\Psi_2, F^- \ \mathsf{act} \vdash \gamma}}{\Psi_1, \Psi_2, F^+ \multimap F^- \ \mathsf{act} \vdash \gamma}$$

4.

(a) By inversion on the rules defining $\rightharpoonup_{\mathsf{ctx}}$, $\rightharpoonup_{\mathsf{conc}}$ the only applicable rule is left activation. Appeal to the induction hypothesis part (3).

(b) The proof is structurally identitcal to part (2b), appealing to part (4a) if there are no active assumptions left in $\Psi$.

(c) Here we proceed by analyzing the top connective of the conclusion, if the conclusion is active. If it is not, we can appeal to the i.h. part (4b).

Case: $\gamma = F_1^+ \multimap F_2^- \ \mathsf{act}$. Hence $C = \mathbf{d}^+ G_1^+ \multimap \mathbf{d}^- G_2^-$. We have $\Xi, G^- \vdash \mathbf{d}^+ G_1^+ \multimap \mathbf{d}^- G_2^-$. It is easy to construct a derivation of $G_1^+, \mathbf{d}^+ G_1^+ \multimap \mathbf{d}^- G_2^- \vdash G_2^-$ by using Lemma 5.1, so by cut we have $\Xi, G_1^+, G^- \vdash G_2^-$. By induction hypothesis (4c) we obtain $\Psi, F_1^+ \ \mathsf{act}, F^- \vdash F_2^- \ \mathsf{act}$. By rule, $\Psi, F^- \vdash F_1^+ \multimap F_2^- \ \mathsf{act}$.

Case: $\gamma = F_1^- \ \& \ F_2^- \ \mathsf{act}$. Hence $C = \mathbf{d}^- G_1^- \ \& \ \mathbf{d}^- G_2^-$. It is easy to construct a derivation of $\mathbf{d}^- G_1^- \ \& \ \mathbf{d}^- G_2^- \vdash G_i^-$ for both $i$ using Lemma 5.1, so by cut we have $\Xi, G^- \vdash G_i^-$ for both $i$. By induction hypothesis (4c) we obtain $\Psi, F^- \vdash F_i^- \ \mathsf{act}$. By rule, $\Psi, F^- \vdash F_1^- \ \& \ F_2^- \ \mathsf{act}$.

Case: $\gamma = \mathbf{d}^- A \ \mathsf{act}$. Here there are two subcases depending on how the translation went.
In the first, $C = \mathbf{d}^- G_0^-$, we know $A$ is of the form $F_0^-$ such that $F_0^- \rightharpoonup^- G_0^-$ for some $G_0^-$. It is easy to construct a derivation of $\mathbf{d}^- G_0^- \vdash G_0^-$ by using Lemma 5.1, so by cut we have $\Xi, G^- \vdash G_0^-$. By induction hypothesis (4c) we obtain $\Psi, F^- \vdash F_0^- \ \mathsf{act}$. By one rule application, $\Psi, F^- \vdash F_0^-$. By another, $\Psi, F^- \vdash \mathbf{d}^- F_0^- \ \mathsf{act}$.
In the second, $C = \mathbf{S}^-(0, B)$ and $A \rightharpoonup B$ for some $A, B$. It is easy to construct a derivation of $\mathbf{S}^-(0, B) \vdash \mathbf{S}_-(0, B)$ so by cut we have $\Xi, G^- \vdash \mathbf{S}_-(0, B)$. By induction hypothesis (4c) we obtain $\Psi, F^- \vdash B$. By rule, $\Psi, F^- \vdash \mathbf{d}^- B \ \mathsf{act}$.

Case: $\gamma = \mathbf{S}^-(n, A)$ $\mathsf{act}$. We know $C = \mathbf{S}^-(n-1, B)$. It is easy to construct a derivation of $\mathbf{S}^-(n+1, B) \vdash \mathbf{S}_-(n+1, B)$ so by cut we have $\Xi, G^- \vdash \mathbf{S}_-(n+1, B)$. By induction hypothesis (4c) we obtain $\Psi, F^- \vdash \mathbf{S}^-(n, B)$ $\mathsf{act}$.

5. Easy division into three cases, depending on the syntactic form of $B_1, B_2$. If they are atoms, we immediately have $p \vdash p$. If $B_i$ are of the form $G_i^+$, then appeal to case (2b) to obtain $F_1^+$ $\mathsf{act} \vdash F_2^+$, and apply the left activation rule to get $F_1^+ \vdash F_2^+$. Otherwise $B_i$ are of the form $G_i^-$, and we appeal to case (4c) to obtain $F_1^- \vdash F_2^-$ $\mathsf{act}$ and apply the right activation rule to obtain $F_1^- \vdash F_2^-$.

■

**Corollary 5.3** *$A \vdash A$ for all $A$.*

**Proof** Follows from part (5) and Lemma 5.1. ■

**Corollary 5.4** *Focussing is complete.*

**Proof** Focussing is just the above deductive system where connectives of the same polarity have been maximally coalesced, and asynchronous connectives are eagerly decomposed. The lemma establishes that we can freely remove $\mathbf{d}$ s that don't change polarity, because under $\rightharpoonup$ they can be mapped identically to $\mathbf{d}$ , just as every other boundary between connectives. Furthermore, by cutting against sequents like $F_1^+$ $\mathsf{act} \vdash F_2^+$ and $F_1^- \vdash F_2^-$ $\mathsf{act}$, we can see that eager activation of asynchronous connectives is complete. ■