A Simple Constructive Proof of Soundness of Labelled Deduction

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1 Introduction

Labelled deduction is a deductive system in which propositions receive labels. Each label may be interpreted as a modal 'world' at which its associated proposition is true. The particular system of labelled deduction that we are concerned with below can be seen from two angles:

1. It is ordinary intuitionistic logic, conservatively modified to enjoy certain properties of classical logic. (e.g., that the disjunction right rule is invertible)

2. It is a judgmental presentation of the Kripke semantics of intuitionistic logic.

To justify the first point of view, it is necessary to show that in fact the things provable by standard intuitionistic logic and the labelled system coincide. The completeness of the labelled system relative to the standard system (that for every standard proof there exists a labelled proof) is relatively straightforward, but the opposite direction is less trivial.

Previous proofs have gone through model-theoretic constructions, exploiting the second point of view above: since the labelled calculus is essentially a transcription of the Kripke semantics, it is easy to show that any proposition that has a labelled proof is forced by every Kripke model. By the completeness theorem for the Kripke semantics of intuitionistic logic, a proposition forced by every model is provable in standard intuitionistic logic.

These proofs are generally (though perhaps not inherently) nonconstructive, and in any event contain an apparently unnecessary detour through model theory. We show below that the detour *is* unnecessary, that there is a constructive and purely syntactic proof of soundness.

2 Labelled Deduction

The sequent calculus of labelled deduction we use is a slight variant of that presented in ???. It is convenient here to use a one-sided sequent, since many func-

tions to be defined below treat hypotheses and conclusions uniformly. Propositions are built from the grammar

$$A, B ::= P \mid A \supset B$$

And contexts from

$$\Gamma ::= \cdot \mid \Gamma, A[p^i]$$

where p is a *label* consisting of a string of letters over some alphabet Σ and i is either $+$ or $-$, indicating whether this proposition is a conclusion or hypothesis, respectively.

The proof rules are

$$\frac{}{\overset{\ell}{\Longrightarrow} \Gamma, A[p^-], A[pq^+]}\, hyp$$

$$\frac{\overset{\ell}{\Longrightarrow} \Gamma, A \supset B[p^-], A[pq] \qquad \overset{\ell}{\Longrightarrow} \Gamma, A \supset B[p^-], B[pq^+]}{\overset{\ell}{\Longrightarrow} \Gamma, A \supset B[p^-]}\, \supset L$$

$$\frac{\overset{\ell}{\Longrightarrow} \Gamma, A[pa^-], B[pa^+]}{\overset{\ell}{\Longrightarrow} \Gamma, A \supset B[p^+]}\, \supset R^a$$

We assume familiarity with the standard natural deduction system for intuitionistic logic, and we write simply $\Gamma \vdash A$ (where Γ is an ordinary unlabelled context of propositions) when such a judgment is provable in that system.

The exchange rule is for granted, and as a consequence we will eventually need to show that the functions we define on contexts are invariant up to permutations of their argument. The implication right rule is parametric in the label a, meaning that a must be fresh in the sense of occuring nowhere else in the context Γ in the sequent $\Longrightarrow^\ell \Gamma, A[pa^-], B[pa^+]$

3 Preliminary Definitions

We define an inequality $\geq$ between labels (which corresponds to the accessibility relation between worlds in Kripke models) by saying $p \geq r$ if r is of the form pq for some string q. The inequality $\geq$ is extended to signed labels by

$$p^i \geq r^j \Leftrightarrow p \geq r \vee (p = r \wedge i = + \wedge j = -)$$

It is important to note that $\geq$ is *decidable*, so that we may deal with its negation without sacrificing the constructive character of the proof. We will need in particular to 'filter' contexts to leave only those things that are (resp. aren't) $\geq$ a particular label.

For $R \in \{\geq, \not\geq\}$ and any signed label $s = p^i$, the expression Rs is a *constraint*. We indicate a context filtered by a constraint by writing the latter as a superscript, e.g. $\Gamma^{\geq s}$ is that part of Γ that contain labels that are $\geq s$.

$$(\Gamma, A[t])^{Rs} = \begin{cases} \Gamma^{Rs}, A^i[t] & \text{if } tRs; \\ \Gamma^{Rs} & \text{otherwise.} \end{cases}$$

$$(\cdot)^{Rs} = \cdot$$

We define a function $\odot^i$ that is a propositional connective associated with the sign i

$$A \odot^+ B = \begin{cases} A & \text{if } B = \bot; \\ A \vee B & \text{otherwise.} \end{cases}$$

$$A \odot^- B = A \supset B$$

This will be used to express the fact that the conclusions in the sequent are a disjunctive collection of possible goals, and the hypotheses function as left-hand sides of implications. The fact that $\odot^+$ 'absorbs' bottom is a nonessential convenience, since $A \vee \bot \dashv\vdash A$ for any A. As a shorthand, we take $\odot^{p^i}$ to mean the same thing as $\odot^i$.

Finally, we define a simple function v that takes a context and returns the disjunction of everything in it:

$$v(A_1[s_1], \ldots, A_n[s_n]) = \bigvee_{i=1}^{n} A_n$$

4 Propositional Interpretation

The goal of the soundness proof is to say that for any world e, if $\Longrightarrow^\ell A[e^+]$, then $\vdash A$. The proof proceeds by strengthening the induction hypothesis to make a claim about every labelled sequent, not just those that have single conclusion.

For each labelled sequent $\Longrightarrow^\ell \Gamma$, we construct a single proposition $[\![\Gamma]\!]$ that internalizes all of the label interactions, and claim that if $\Longrightarrow^\ell \Gamma$, then $\vdash [\![\Gamma]\!]$. For as we proceed up from the final inference of a proof, fresh worlds will be introduced and manipulated; we need a way of saying in simpler terms what a complex labelled sequent ultimately means.

It should be noted that the key idea here is no different than the one that drives the (very easy) proof of soundness of a system intermediate between the labelled and standard systems, the multiple conclusion inutitionistic sequent calculus. The induction hypothesis there is that if $\Gamma \Longrightarrow^m \Delta$, then $\Gamma \vdash \bigvee \Delta$; the comma on the right is internalized as disjunction, and the content of the proof is showing that the way the right-hand context is manipulated by the source language rules is consistent with such an internalization.

This choice of internalization means that, when one comes to the or-right-rule case in the proof of soundness, that case is handled trivially. For the rule itself relates the judgmental notion of comma-on-the-right to the thing we have chosen to internalize it as, disjunction. We similarly choose an interpretation for the labelled calculus so that the application of the implication right rule leaves the interpretation unchanged, up to provable equivalence.

The present complication is only that internalizing worlds is somewhat more complicated than simply giving the connective that comma-on-the-right is a proxy for. Labels express implications and disjunctions, but only in a way that arises out of their relationship to one another.

An intuition for how the present translation proceeds is as follows: imagine that we are trying to prove

$$\stackrel{\ell}{\Longrightarrow}(A \vee B) \supset ((C \supset E) \vee (D \supset F))[e^+]$$

We will arrive at the sequent

$$\stackrel{\ell}{\Longrightarrow} A \vee B[e^-], C[ea^-], D[eb^-], E[ea^+], F[eb^+]$$

if we simply invert the implies-right and or-right rules. Consider how we would internalize this latter sequent.

There are three worlds represented here, e, ea, and eb. The latter two are $\geq e$ (one might also say accessible from e, or in the 'future' of e) and so 'have access' to any assumptions made at e, (specifically $A \vee B$) this intuition being drawn from the monotonicity property of Kripke models. The conclusions $E[ea^+], F[eb^+]$ each have access to one further assumption that occurs at its associated world. The conclusion E 'knows' that C holds, and F knows that D.

At a high level, we are saying that if all the assumptions $A \vee B[e^-], C[ea^-]$, and $D[eb^-]$ hold, then *some* conclusion can be satisfied. Therefore we construct a proposition as a tree where the nodes of arity ≥ 1 are essentially disjunctive. We begin with the earliest world e, and examine what hypotheses are available there; just $A \vee B$. Our tree looks like

$$(A \supset B) \supset [\ldots \text{information at world } e \ldots]$$

we consider all future worlds of e (ea and eb) and reason that of all the goals that might be satisfied, they might be satisfied in either a world in the future of ea, or one in the future of eb. Thus we expand the tree to

$$(A \supset B) \supset ([\ldots \text{information at world } ea \ldots] \vee$$

$$[\ldots \text{information at world } eb \ldots])$$

We know what hypotheses are available at these worlds, though, just by inspecting the context. We have then

$$(A \supset B) \supset (C \supset [\ldots \text{conclusions at world } ea \ldots] \vee$$

$$(D \supset [\ldots \text{conclusions at world } eb \ldots])$$

And finally we know what conclusions are available at each world, and we end up with

$$(A \vee B) \supset ((C \supset E) \vee (D \supset F))$$

precisely the proposition we started with.

In general we produce an expression of disjunctions and implications of propositions occurring in the original sequent, in such a way that propositions at earlier worlds are in positions in the expression tree above propositions at later worlds.

The formal definition of the interpretation $[\![\Gamma]\!]$ of Γ requires still a slight further generalization; we define a function $[\![\Gamma]\!](\Delta)$ on two contexts Γ and Δ, where Δ represents 'postponed' work arising from earlier Γs. The desired function $[\![\Gamma]\!]$ is the result of passing an empty context as the second argument $[\![\Gamma]\!](\cdot)$. Although the intuition for the definition proceeds 'recursively over worlds,' building up a proposition by starting at labels early in the $\leq$-ordering and moving to later ones, the definition is recursive over the structure of contexts. This is necessary in the subsequent proof because the inductive hypothesis is that the interpretation of some sequent is provable. Therefore we need to be able to recursively compute (in terms of the interpretation of Γ) the interpretation of a sequent such as, for example, the first premise of the implication left rule, $\Longrightarrow^{\ell} \Gamma, A \supset B[p^-], A[pq^+]$ that consists of a context extended by some further propositions.

The definition is this:

$$[\![\cdot]\!](\Delta) = v(\Delta)$$
$$[\![\Gamma, A[s]]\!](\Delta) = [\![\Gamma^{\not\geq s}]\!](\Delta^{\not\geq s}, A \odot^s ([\![\Gamma^{\geq s}]\!](\Delta^{\geq s}))[s])$$

If we encounter the context $\Gamma, A[s]$, we wish to take the interpretation of Γ, and insert the proposition A in its proper place. We therefore divide Γ (and Δ) into two parts, the $\geq s$ part and the $\not\geq s$ part. We then attach A to the $\geq s$ part via either a disjunction or implication according to whether A occurred positively or negatively, and we save the result $A \odot^s ([\![\Gamma^{\geq s}]\!](\Delta^{\geq s}))$ marked as belonging to position s in the Δ of the recursive call.

Our goal is now to prove

Theorem 4.1 *If* $\Longrightarrow^{\ell} \Gamma$*, then* $\vdash [\![\Gamma]\!](\cdot)$.

from which it follows that

Corollary 4.2 *If* $\Longrightarrow^{\ell} A[e^+]$*, then* $\vdash A$.

Proof By the main theorem, we have $\vdash [\![A[e^+]]\!](\cdot)$, but

$$\begin{aligned}[\![A[e^+]]\!](\cdot) &= [\![\cdot]\!](A \odot^+ [\![\cdot]\!](\cdot)[e^+]) \\ &= [\![\cdot]\!](A \odot^+ \bot[e^+]) \\ &= [\![\cdot]\!](A[e^+]) \\ &= A\end{aligned}$$

So we have $\vdash A$. ■

Before we get to the main soundness proof we establish two substantial lemmas, one that allows us to transfer properties of $\supset, \vee$ to propositional functions arising from $[\![\Gamma]\!](\Delta)$, and the other that $[\![\Gamma]\!](\Delta)$ is invariant (up to $\dashv\vdash$) under permutations of Γ.

5 Lifting

The first we treat is the transfer lemma. First, some examples of what it is used for:

An important property of the interpretation function (which we will have shown by the end of this section) is that it is monotone[1] in a proposition appearing in Δ, i.e.

$$\text{If } A \vdash B, \text{ then } [\![\Gamma]\!](A[s]) \vdash [\![\Gamma]\!](B[s])$$

We can see that 'basic' propositional functions are monotone in the same sense: $\lambda X.C \vee X$ and $\lambda X.C \supset X$:

Fact 5.1

- *If* $A \vdash B$, *then* $C \vee A \vdash C \vee B$
- *If* $A \vdash B$, *then* $C \supset A \vdash C \supset B$

We will also eventually need a to use the result that

$$[\![\Gamma]\!](A[s]) \wedge [\![\Gamma]\!](B[s]) \vdash [\![\Gamma]\!](A \wedge B[s]) \qquad (*)$$

and we observe that a similar property holds of our 'basic' functions:

Fact 5.2

- *If* $(C \vee A) \wedge (C \vee B) \vdash C \vee (A \wedge B)$
- *If* $(C \supset A) \wedge (C \supset B) \vdash C \supset (A \wedge B)$

It turns out that we can argue that $\lambda X.[\![\Gamma]\!](X[s])$ is merely built from copies of the basic functions $\lambda X.C \vee X$ and $\lambda X.C \supset X$, in such a way that both of these situations are instances of a more general one: every property of a certain type that holds of the basic functions, also holds of the interpretation function.

For observe that in both situations above that there is a claim of provability in which uniformly either one of the basic functions, or else $\lambda X.[\![\Gamma]\!](X[s])$, is used. Let F and G be functions of type $(prop \to prop) \to prop$, i.e. each takes a function such as $\lambda X.C \supset X$ from propositions to propositions, and returns a proposition. The relations $\triangleright$ and $\blacktriangleright$ are defined as follows:

Definition

$$FG \triangleright f \Leftrightarrow F(f) \vdash G(f)$$

$$FG \blacktriangleright f \Leftrightarrow (FG \triangleright g \text{ implies } FG \triangleright fg \text{ for all } g : prop \to prop)$$

[1] Despite even negative sign annotations, which might suggest that such positions are contravariant. We could have instead made the issue explicit, and said that Δ is a different syntactic sort of context than Γ to emphasize that it contains essentially only positive occurrences, for for simplicity we did not.

The relation $\triangleright$ is the main thing we are concerned with; When $F(f) = f(A) \wedge f(B)$ and $G(f) = f(A \wedge B)$, then Fact 5.2 is essentially the fact that $FG \triangleright \lambda X.C \odot^i X$, and the result $(*)$ it the claim that $FG \triangleright \lambda X.[\![\Gamma]\!](X[s])$. However we actually need a the slight strengthening $\blacktriangleright$ of $\triangleright$ to permit a proof of the following result:

Lemma 5.3 (Lifting) *Suppose $\lambda X.FG \blacktriangleright C \odot^i X$ for any C, i. Then*

$$FG \blacktriangleright \lambda X.[\![\Gamma]\!](\Delta_1, X[s], \Delta_2)$$

for any $\Gamma, \Delta_1, \Delta_2$.

Proof By induction on Γ.

We often write $\Box$ for the argument of an anonymous function instead of using λs everywhere. Observe that because $FG \blacktriangleright C \odot^+ \Box$, we also have $FG \blacktriangleright \Box \vee C$ and $FG \blacktriangleright C \vee \Box$ for any C.

If Γ is empty, then we need to show

$$FG \blacktriangleright A_1 \vee (\cdots (A_n \vee (\Box \vee v(\Delta_2))) \cdots)$$

assuming $\Delta_1 = A_1[s_1], \ldots, A_n[s_n]$. In other words our goal is

$$FG \blacktriangleright (A_1 \vee \Box) \circ \cdots \circ (A_n \vee \Box) \circ (\Box \vee v(\Delta_2))$$

Let g be given, and assume $FG \triangleright g$. We must show that

$$FG \triangleright (A_1 \vee \Box) \circ \cdots \circ (A_n \vee \Box) \circ (\Box \vee v(\Delta_2)) \circ g$$

but this follows by repeatedly using the fact that $FG \blacktriangleright C \vee \Box$ and $FG \blacktriangleright \Box \vee C$.

Otherwise, the context is nonempty and is of the form $\Gamma, C[t]$. Again we let g be given and assume $FG \triangleright g$. Our goal is to show

$$FG \triangleright ([\![\Gamma, C[t]]\!](\Delta_1, (\Box)[s], \Delta_2)) \circ g$$

There are two cases depending on the relationship between s and t.

Case: $s \geq t$. Then we must show

$$FG \triangleright ([\![\Gamma^{\not\geq t}]\!](\Delta_1^{\not\geq t}, \Delta_2^{\not\geq t}, C \odot^t ([\![\Gamma^{\geq t}]\!](\Delta_1^{\geq t}, \Box[s], \Delta_2^{\geq t}))[t]) \circ g$$

or in other words that $FG \triangleright f_1 \circ f_2 \circ f_3 \circ g$ for

$$f_1 = [\![\Gamma^{\not\geq t}]\!](\Delta_1^{\not\geq t}, \Delta_2^{\not\geq t}, \Box[t])$$

$$f_2 = C \odot^t \Box$$

$$f_3 = [\![\Gamma^{\geq t}]\!](\Delta_1^{\geq t}, \Box[s], \Delta_2^{\geq t})$$

which follows by two applications of the induction hypothesis (for f_1, f_3) and a use of the assumption that $FG \blacktriangleright C \odot^t \Box$ for f_2.

Case: $s \not\geq t$. Then we must show

$$FG \triangleright ([\![\Gamma^{\not\geq t}]\!](\Delta_1^{\not\geq t}, \Box[s], \Delta_2^{\not\geq t}, C \odot^t ([\![\Gamma^{\geq t}]\!](\Delta_1^{\geq t}, \Delta_2^{\geq t}))[t]) \circ g$$

but this follows directly from the induction hypothesis.

■

From this we can extract the following results:

Corollary 5.4 *If $FG \blacktriangleright \lambda X.C \odot^i X$ for any C, i, and $FG \triangleright \lambda X.X$, then $FG \triangleright [\![\Gamma]\!](\Box[s])$ for any Γ.*

Lemma 5.5 *The following pairs of functions enjoy the properties in the previous corollary:*

1. $F(f) = A, G(f) = f(A)$
2. $F(f) = f(A) \wedge f(B), G(f) = f(A \wedge B)$
3. $F(f) = f(A \supset B), G(f) = A \supset f(B)$
4. $F(f) = f(A), G(f) = f(B)$ *(if $A \vdash B$)*

Proof An easy exercise in directly constructing small proofs in ordinary intuitionistic logic. For example, let us do case 3. We need to show that, for any g,

- If $g(A \supset B) \vdash A \supset g(B)$, then $C \supset g(A \supset B) \vdash A \supset (C \supset g(B))$.
- If $g(A \supset B) \vdash A \supset g(B)$, then $C \vee g(A \supset B) \vdash A \supset (C \vee g(B))$.
- $A \supset B \vdash A \supset B$.

But these are all trivial to show. ■

Corollary 5.6

1. $A \vdash [\![\Gamma]\!](A[s])$
2. $[\![\Gamma]\!](A[s]) \wedge [\![\Gamma]\!](B[s]) \vdash [\![\Gamma]\!](A \wedge B[s])$
3. $[\![\Gamma]\!](A \supset B[s]) \vdash A \supset [\![\Gamma]\!](B[s])$
4. $[\![\Gamma]\!](A_1[s_1], \ldots, A_n[s_n]) \vdash [\![\Gamma]\!](B_1[s_1], \ldots, B_n[s_n])$ *(if $A_k \vdash B_k$)*

6 Exchange

The second main lemma is showing that the interpretation function is compatible with the exchange rule. Being able to permute Δ is easy: it is just a straightforward induction on the definition of $[\![\Gamma]\!](\Delta)$, whose base case takes advantage of the commutativity (up to $\dashv\vdash$) of $\vee$.

Lemma 6.1 *If Δ' is a permutation of Δ, then $[\![\Gamma]\!](\Delta) \vdash [\![\Gamma]\!](\Delta')$.*

Proof By induction on Γ. ■

To show that Γ can also be permuted, the step is to show that we can 'drill down' into a pair of contexts that have a common suffix, and then from there interchange two adjacent propositions. Being able to swap any pair of propositions anywhere within the context, we can achieve any permutation.

We will need to consider lists of constraints $L ::= R_1 s_1, \ldots, R_n s_n$ whose effect on a context is the evident cumulative filtering that extracts everything that satisfies every constraint in the list.

$$\Gamma^{R_1 s_1, \ldots, R_n s_n} = (\cdots(\Gamma^{R_1 s_1})\cdots)^{R_n s_n}$$

For the sake of the implication right rule case of the final proof, we consider one-place predicates P on signed labels. If P is such a predicate, we extend it to constraint lists by requiring that every signed label in the constraint satisfies P, and to contexts in the same fashion.

$$P(R_1 s_1, \ldots, R_n s_n) \Leftrightarrow \forall i. P(s_i)$$

$$P(A_1[s_1], \ldots, A_n[s_n]) \Leftrightarrow \forall i. P(s_i)$$

Now make the following definition:

Definition Let P be a predicate on signed labels. Then $\Gamma_1 \rightsquigarrow^P \Gamma_2$ is a binary relation between labelled contexts that is defined to hold iff $[\![\Gamma_1^L]\!](\Delta) \vdash [\![\Gamma_2^L]\!](\Delta)$ for all L, Δ such that $P(L)$, $P(\Delta)$.

Lemma 6.2 *If $\Gamma_1 \rightsquigarrow^P \Gamma_2$ and $P(\Gamma)$, then $\Gamma_1, \Gamma \rightsquigarrow^P \Gamma_2, \Gamma$.*

Proof By induction on Γ. The base case is trivial. The induction step is showing that if $\Gamma_1 \rightsquigarrow^P \Gamma_2$ and $P(s)$, then $\Gamma_1, A[s] \rightsquigarrow^P \Gamma_2, A[s]$. Let suitable L, Δ be given. We must show $[\![(\Gamma_1, A[s])^L]\!](\Delta) \vdash [\![(\Gamma_2, A[s])^L]\!](\Delta)$. Let (for $m, n \in \{1, 2\}$)

$$P_{mn} = [\![\Gamma_m^{L, \not\geq s}]\!](\Delta^{\not\geq s}, A \odot^s ([\![\Gamma_n^{L, \geq s}]\!](\Delta^{\geq s})[s])$$

If $A[s]$ is erased by L, then we are already done. Otherwise we need to show $P_{11} \vdash P_{22}$. By induction hypothesis, $[\![\Gamma_1^{L, \geq s}]\!](\Delta^{\geq s}) \vdash [\![\Gamma_2^{L, \geq s}]\!](\Delta^{\geq s})$ hence by Lemma 5.6 part 4 we have $P_{11} \vdash P_{12}$. By induction hypothesis again, $P_{12} \vdash P_{22}$, and we are done. ■

Now we can establish that any two propositions on the right can be swapped:

Lemma 6.3 $\Gamma, A_0[s_0], A_1[s_1] \leftrightsquigarrow^P \Gamma, A_1[s_1], A_0[s_0]$

Proof Let suitable L, Δ be given. Assume without loss that L doesn't erase either $A_0[s_0]$ or $A_1[s_1]$. There are essentially three possible relationships between s_0 and s_1.

Case: $s_0 \geq s_1$ and $s_1 \not\geq s_0$. (We might say that $s_0 > s_1$; the case where $s_1 > s_0$ is symmetric) Expanding out definitions on the left, (using only the fact that $s_0 \geq s_1$) we get

$$[\![\Gamma^{L,\not\geq s_1}]\!](\Delta^{\not\geq s_1}, A_1 \odot^{s_1} ([\![\Gamma^{L,\geq s_1}, A_0[s_0]]\!](\Delta^{\geq s_1}))[s_1]) =$$

$$[\![\Gamma^{L,\not\geq s_1,\not\geq s_0}]\!](\Delta^{\not\geq s_1,\not\geq s_0}, A_1 \odot^{s_1}$$
$$([\![\Gamma^{L,\geq s_1,\not\geq s_0}]\!](\Delta^{\geq s_1,\not\geq s_0}, A_0 \odot^{s_0}$$
$$([\![\Gamma^{L,\geq s_1,\geq s_0}]\!](\Delta^{\geq s_1,\geq s_0}))[s_0]))[s_1])$$

And expanding on the right (using both that $s_0 \geq s_1$ and $s_1 \not\geq s_0$) we get

$$[\![\Gamma^{L,\not\geq s_0}, A_1[s_1]]\!](\Delta^{\not\geq s_0}, A_0 \odot^{s_0} ([\![\Gamma^{L,\geq s_0}]\!](\Delta^{\geq s_0}))[s_0])$$

$$= [\![\Gamma^{L,\not\geq s_0,\not\geq s_1}]\!](\Delta^{\not\geq s_0,\not\geq s_1}, A_1 \odot^{s_1}$$
$$([\![\Gamma^{L,\not\geq s_0,\geq s_1}]\!](\Delta^{\not\geq s_0,\geq s_1}, A_0 \odot^{s_0}$$
$$([\![\Gamma^{L,\geq s_0,\geq s_1}]\!](\Delta^{\geq s_0,\geq s_1}))[s_0]))[s_1])$$

These expressions differ only in the order of application of constraints, so we are done.

Case: $s_0 = s_1$. Say both are equal to s. We can reuse our reasoning about the left-hand side from the previous case, and go further in this case because we know that $\Gamma^{\geq s_1,\not\geq s_0}$ is necessarily empty, and so too with $\Delta^{\geq s_1,\not\geq s_0}$. Hence it becomes

$$[\![\Gamma^{L,\not\geq s}]\!](\Delta^{\not\geq s}, A_1 \odot^{s} (A_0 \odot^{s} ([\![\Gamma^{L,\geq s}]\!](\Delta^{\geq s})))[s])$$

and the right-hand side is just

$$[\![\Gamma^{L,\not\geq s}]\!](\Delta^{\not\geq s}, A_0 \odot^{s} (A_1 \odot^{s} ([\![\Gamma^{L,\geq s}]\!](\Delta^{\geq s})))[s])$$

which follows from Lemma 5.6 part 4 and the fact that

$$A \odot^i (B \odot^i C) \dashv\vdash B \odot^i (A \odot^i C)$$

for any A, B, C, i.

Case: $s_0 \perp s_1$, that is, $s_0 \not\geq s_1$ and $s_1 \not\geq s_0$. Here when we expand the left-hand side (taking advantage of a consequence of the definition of $\geq$; that if $s_0 \perp s_1$ and $s \geq s_0$, then $s \not\geq s_1$) we get

$$\begin{aligned}
&[\![\Gamma^{L,\not\geq s_1}, A_0[s_0]]\!](\Delta^{\not\geq s_1}, A_1 \odot^{s_1} ([\![\Gamma^{L,\geq s_1}]\!](\Delta^{\geq s_1}))[s_1]) \\
&\quad = [\![\Gamma^{L,\not\geq s_1,\not\geq s_0}]\!](\Delta^{\not\geq s_1,\not\geq s_0}, \\
&\quad A_1 \odot^{s_1} ([\![\Gamma^{L,\geq s_1}]\!](\Delta^{\geq s_1}))[s_1], \\
&\quad A_0 \odot^{s_0} ([\![\Gamma^{L,\geq s_0}]\!](\Delta^{\geq s_0}))[s_0])
\end{aligned}$$

and on the right-hand side we get the same thing with 0 and 1 interchanged. By Lemma 6.1 they are provably equivalent.

■

Corollary 6.4 *Suppose Γ' is a permutation of Γ. Then $\Gamma \leftrightsquigarrow^P \Gamma'$.*

Lemma 6.5 *Let $a \in \Sigma$ be given. Put $P(w) \Leftrightarrow w \not\geq pa$. Then*

$$A[pa^-], B[pa^+] \leftrightsquigarrow^P A \supset B[p^+]$$

Proof No constraint that satisfies P can discriminate p from pa, so we assume without loss that L erases nothing. Taking advantage of the fact that nothing $\geq pa$ occurs in Δ, we know the goal is to show that

$$[\![A[pa^-], B[pa^+]]\!](\Delta) = v(\Delta, A \supset B[pa])$$

is provably equivalent to

$$[\![A \supset B[p^+]]\!](\Delta) = v(\Delta^{\not\geq p}, (A \supset B) \vee v(\Delta^{\geq p})[p])$$

but this follows immediately from the commutativity of $\vee$. ■

7 Soundness

Theorem 7.1 *If $\Longrightarrow^\ell \Gamma$, then $\vdash [\![\Gamma]\!](\cdot)$.*

Proof By induction on the derivation of $\Longrightarrow^\ell \Gamma$.

Case:

$$\frac{}{\overset{\ell}{\Longrightarrow} \Gamma, A[p^-], A[pq^+]}\ hyp$$

Expanding out the definitions, we must show

$$\vdash [\![\Gamma^{\not\geq p}]\!](A \supset [\![\Gamma^{\not\geq pq,\geq p}]\!](A \vee [\![\Gamma^{\geq pq}]\!](\cdot)[pq])[p])$$

Using Lemma 5.6 parts 1 and 4 we find

$$A \supset (A \vee X) \vdash A \supset [\![\Gamma^{\not\geq pq,\geq p}]\!](A \vee X[pq])$$

for $X = [\![\Gamma^{\geq pq}]\!](\cdot)$, and Lemma 5.6 parts 1 and 4 once more on the outside yields

$$A \supset (A \vee X) \vdash [\![\Gamma^{\not\geq p}]\!](A \supset [\![\Gamma^{\not\geq pq, \geq p}]\!](A \vee X[pq])[p])$$

but obviously we can prove that $A \supset (A \vee X)$, so we are done.

Case:

$$\frac{\overset{\ell}{\Longrightarrow} \Gamma, A \supset B[p^-], A[pq] \qquad \overset{\ell}{\Longrightarrow} \Gamma, A \supset B[p^-], B[pq^+]}{\overset{\ell}{\Longrightarrow} \Gamma, A \supset B[p^-]} \supset L$$

By induction hypothesis we know

$$\vdash [\![\Gamma, A \supset B[p^-], A[pq^+]]\!](\cdot)$$

$$\vdash [\![\Gamma, A \supset B[p^-], B[pq^-]]\!](\cdot)$$

Expanding out definitions we have

$$\vdash [\![\Gamma^{\not\geq p}]\!]((A \supset B) \supset [\![\Gamma^{\not\geq pq, \geq p}]\!](A \vee [\![\Gamma^{\geq pq}]\!](\cdot)[pq])[p])$$

$$\vdash [\![\Gamma^{\not\geq p}]\!]((A \supset B) \supset [\![\Gamma^{\not\geq pq, \geq p}]\!](B \supset [\![\Gamma^{\geq pq}]\!](\cdot)[pq])[p])$$

Using Lemma 5.6 parts 2 and 4 we get

$$\vdash [\![\Gamma^{\not\geq p}]\!]((A \supset B) \supset [\![\Gamma^{\not\geq pq, \geq p}]\!]((A \vee X) \wedge (B \supset X)[pq])[p]) \qquad (*)$$

for $X = [\![\Gamma^{\geq pq}]\!](\cdot)[pq]$. It is easy to show that

$$(A \vee X) \wedge (B \supset X) \vdash (A \supset B) \supset X$$

so by using Lemma 5.6 parts 3 and 4, (and writing $F(X)$ in place of $[\![\Gamma^{\not\geq pq, \geq p}]\!](X[pq])$) we can see

$$F((A \vee X) \wedge (B \supset X)) \vdash F((A \supset B) \supset X) \vdash (A \supset B) \supset F(X)$$

hence Lemma 5.6 part 4 and fact that $C \supset (C \supset D) \vdash C \supset D$ for any C, D give us

$$(A \supset B) \supset F((A \vee X) \wedge (B \supset X)) \vdash$$

$$(A \supset B) \supset (A \supset B) \supset F(X) \vdash (A \supset B) \supset F(X)$$

hence by Lemma 5.6 part 4 one final time we find

$$[\![\Gamma^{\not\geq p}]\!]((A \supset B) \supset F((A \vee X) \wedge (B \supset X))[p]) \vdash [\![\Gamma^{\not\geq p}]\!]((A \supset B) \supset F(X)[p])$$

which can be cut against $(*)$ to produce the required result.

Case:

$$\frac{\overset{\ell}{\Longrightarrow} \Gamma, A[pa^-], B[pa^+]}{\overset{\ell}{\Longrightarrow} \Gamma, A \supset B[p^+]} \supset R^a$$

By induction hypothesis (here taking more obvious advantage of Corollary 6.4) we know

$$\vdash [\![A[pa^-], B[pa^+], \Gamma]\!](\cdot)$$

and we must show

$$\vdash [\![A \supset B[p^+], \Gamma]\!](\cdot)$$

By Lemma 6.5 we know that

$$A[pa^-], B[pa^+] \vdash^P A \supset B[p^+]$$

for $P(w) \Leftrightarrow w \not\geq pa$. Now $P(\Gamma)$, because the label a is fresh. So Lemma 6.2 implies that

$$A[pa^-], B[pa^+], \Gamma \vdash^P A \supset B[p^+], \Gamma$$

from which we can extract the special case of

$$[\![A[pa^-], B[pa^+], \Gamma]\!](\cdot) \vdash [\![A \supset B[p^+], \Gamma]\!](\cdot)$$

and we are done.

■