$$\frac{}{\Gamma, A[p] \vdash_\ell A[pq], \Delta}$$

$$\frac{\Gamma, A \supset B[p] \vdash_\ell A[pq], \Delta \qquad \Gamma, A \supset B[p], B[pq] \vdash_\ell \Delta}{\Gamma, A \supset B[p] \vdash_\ell \Delta}$$

$$\frac{\Gamma, A[pa] \vdash_\ell B[pa], \Delta}{\Gamma \vdash_\ell A \supset B[p], \Delta}$$

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1 Soundness of Labelled Sequent Calculus

The basic inequality $\geq$ between worlds is defined by saying $p \geq r$ if r is of the form pq. We have a notion of *signed worlds* p^+ or p^- (for which we use the variables s, t) that have an annotation indicating whether they came from a positive or negative position in the context. Essentially we work with a one-sided sequent. The inequality $\geq$ is extended to signed worlds by saying

$$p^i \geq r^j \Leftrightarrow p \geq r \vee (p = r \wedge i = + \wedge j = -)$$

For $R \in \geq, \not\geq$ and any signed world s, the expression Rs is a *constraint*. We indicate a context filtered out by a constraint by writing the latter as a superscript, e.g. $\Gamma^{\geq s}$ is that part of Γ in the future of world s.

$$(\Gamma, A[t])^{Rs} = \begin{cases} \Gamma^{Rs}, A^i[t] & \text{if } tRs; \\ \Gamma^{Rs} & \text{otherwise.} \end{cases}$$

$$(\cdot)^{Rs} = \cdot$$

The central definition of the function $[\![\Gamma]\!](\Delta)$ — which realizes sequents as propositions — is as follows:

$$A \odot^{w^+} B = \begin{cases} A & \text{if } B = \bot; \\ A \vee B & \text{otherwise.} \end{cases}$$

$$A \odot^{w^-} B = A \supset B$$

$$[\![\Gamma, A[s]]\!](\Delta) = [\![\Gamma^{\not\geq s}]\!](\Delta^{\not\geq s}, A \odot^s ([\![\Gamma^{\geq s}]\!](\Delta^{\geq s}))[s])$$

$$[\![\cdot]\!](\Delta) = v(\Delta)$$

$$v(A_1[s_1], \ldots, A_n[s_n]) = \bigvee_{i=1}^{n} A_n$$

Just to be definite about the meaning of $\bigvee$, say it is defined by

$$\bigvee_{i=1}^{n} A_n = (\cdots(\cdots) \vee A_2) \vee A_1$$

This peculiar choice of reversing the list is motivated by the proof of the inductive base case in Lemmas 1.2 and 1.3.

Definition A binary relation $\rightsquigarrow$ on propositions is called *monotone* if

- $A \supset B \rightsquigarrow A \supset B'$ for any $B \rightsquigarrow B'$
- $B \wedge C \rightsquigarrow B' \wedge C'$ for any $B \rightsquigarrow B'$ and $C \rightsquigarrow C'$
- $B \vee C \rightsquigarrow B' \vee C'$ for any $B \rightsquigarrow B'$ and $C \rightsquigarrow C'$

Definition Δ is said to be *outside* Γ if $s \not\geq t$ for every $A[s] \in \Gamma, B[t] \in \Delta$.

Lemma 1.1 *Suppose $\rightsquigarrow$ is monotone. Let $\Delta^k = A_1^k[s_1], \dots, A_n^k[s_n]$ for $k \in \{1, 2\}$. Then*

$$[\![\Gamma]\!](\Delta^1) \rightsquigarrow [\![\Gamma]\!](\Delta^2)$$

for any Γ such that the context Δ^1, Δ^2 are both outside Γ, and $A_m^1 \rightsquigarrow A_m^2$ for all $m \in 1 \dots n$.

Proof By induction on Γ. ■

Definition A binary relation $\rightsquigarrow$ on propositions is called *swappable* if

$$A \odot^s (B \odot^s C) \rightsquigarrow B \odot^s (A \odot^s C)$$

for any A, B, C, s. It is *strongly swappable* if also $B \vee (A \supset C) \rightsquigarrow A \supset (B \vee C)$ for any A, B, C, s.

Lemma 1.2 *Suppose $\rightsquigarrow$ is a monotone, strongly swappable preorder. Then*

$$[\![\Gamma]\!](A \supset B[s], \Delta) \rightsquigarrow A \supset [\![\Gamma]\!](B[s], \Delta)$$

for any Γ such that $B[s], \Delta$ is outside Γ.

Proof By induction on Γ.

At the base case, where Γ is empty, suppose that the context Δ has the form $D_1[s_1], \dots, D_n[s_n]$. we must show

$$(\bigvee_{i=1}^{n} D_n \vee (A \supset B)) \rightsquigarrow A \supset (\bigvee_{i=1}^{n} D_n \vee B)$$

but this is just an instance of strong swappability.

The induction step is when Γ is of the form $\Gamma, C[t]$. If $s \geq t$, then we are to show

$$\begin{gathered}[\![\Gamma^{\not\geq t}]\!](\Delta^{\not\geq t}, C \odot^t ([\![\Gamma^{\geq t}]\!]((A \supset B)[s], \Delta^{\geq t}))) \\ \rightsquigarrow A \supset [\![\Gamma^{\not\geq t}]\!](\Delta^{\not\geq t}, C \odot^t ([\![\Gamma^{\geq t}]\!](B[s], \Delta^{\geq t})))\end{gathered}$$

which is possible by appeal to the induction hypothesis, one direct use of the assumption of strong swappability to push the A left past C, and liberal use of monotonicity and the preorder axioms. If $s \not\geq t$, then we are to show

$$\begin{gathered}[\![\Gamma^{\not\geq t}]\!]((A \supset B)[s], \Delta^{\not\geq t}, C \odot^t ([\![\Gamma^{\geq t}]\!](\Delta^{\geq t}))) \\ \rightsquigarrow A \supset [\![\Gamma^{\not\geq t}]\!](B[s], \Delta^{\not\geq t}, C \odot^t ([\![\Gamma^{\geq t}]\!](\Delta^{\geq t})))\end{gathered}$$

which follows from the induction hypothesis. ■

Definition A binary relation $\rightsquigarrow$ on propositions is called *lex* if

$$(C \odot^s A) \wedge (C \odot^s B) \rightsquigarrow C \odot^s (A \wedge B)$$

for any A, B, C, s

Lemma 1.3 *Suppose $\rightsquigarrow$ is a monotone, lex preorder. Then*

$$[\![\Gamma]\!](A[s], \Delta) \wedge [\![\Gamma]\!](B[s], \Delta) \rightsquigarrow [\![\Gamma]\!](A \wedge B[s], \Delta)$$

for any Γ such that $A \wedge B[s], \Delta$ is outside Γ.

Proof By induction on Γ.

At the base case, where Γ is empty, suppose that the context Δ has the form $D_1[s_1], \ldots, D_n[s_n]$. we must show

$$(\bigvee_{i=1}^{n} D_n \vee A) \wedge (\bigvee_{i=1}^{n} D_n \vee B) \rightsquigarrow (\bigvee_{i=1}^{n} D_n \vee (A \wedge B))$$

but this is just an instance of lexity.

The induction step is when Γ is of the form $\Gamma, C[t]$. If $s \geq t$, then we are to show

$$\begin{gathered}[\![\Gamma^{\not\geq t}]\!](\Delta^{\not\geq t}, C \odot^t ([\![\Gamma^{\geq t}]\!](A[s], \Delta^{\geq t}))) \\ \wedge [\![\Gamma^{\not\geq t}]\!](\Delta^{\not\geq t}, C \odot^t ([\![\Gamma^{\geq t}]\!](B[s], \Delta^{\geq t}))) \\ \rightsquigarrow [\![\Gamma^{\not\geq t}]\!](\Delta^{\not\geq t}, C \odot^t ([\![\Gamma^{\geq t}]\!](A \wedge B[s], \Delta^{\geq t})))\end{gathered}$$

which is possible by appeal to the induction hypothesis, one direct use of the assumption of strong swappability to push the closing zipper left past C, and liberal use of monotonicity and the preorder axioms. If $s \not\geq t$, then we are to show

$$\begin{gathered}[\![\Gamma^{\not\geq t}]\!](A[s], \Delta^{\not\geq t}, C \odot^t ([\![\Gamma^{\geq t}]\!](\Delta^{\geq t}))) \\ \wedge [\![\Gamma^{\not\geq t}]\!](B[s], \Delta^{\not\geq t}, C \odot^t ([\![\Gamma^{\geq t}]\!](\Delta^{\geq t}))) \\ \rightsquigarrow [\![\Gamma^{\not\geq t}]\!](A \wedge B[s], \Delta^{\not\geq t}, C \odot^t ([\![\Gamma^{\geq t}]\!](\Delta^{\geq t})))\end{gathered}$$

which follows from the induction hypothesis. ■

Definition A binary relation $\rightsquigarrow$ on propositions is called *weakenable* if $A \rightsquigarrow (C \odot^s A)$ for any A, C, s.

Lemma 1.4 *Suppose $\rightsquigarrow$ is a monotone, weakenable preorder. Then*

$$A \rightsquigarrow [\![\Gamma]\!](A[s], \Delta)$$

for any Γ such that $A[s], \Delta$ is outside Γ.

Proof By induction on Γ.

At the base case, where Γ is empty, suppose that the context Δ has the form $D_1[s_1], \ldots, D_n[s_n]$. we must show

$$A \rightsquigarrow (\bigvee_{i=1}^{n} D_n \vee A)$$

but this is just an instance of weakenability.

The induction step is when Γ is of the form $\Gamma, C[t]$. If $s \geq t$, then we are to show

$$A \rightsquigarrow [\![\Gamma^{\not\geq t}]\!](\Delta^{\not\geq t}, C \odot^t ([\![\Gamma^{\geq t}]\!](A[s], \Delta^{\geq t})))$$

which is possible by appeal to the induction hypothesis, one direct use of the assumption of weakenability to add the $C\odot^t$ to the right-hand side of $\rightsquigarrow$, and liberal use of monotonicity and the preorder axioms. If $s \not\geq t$, then we are to show

$$A \rightsquigarrow [\![\Gamma^{\not\geq t}]\!](A[s], \Delta^{\not\geq t}, C \odot^t ([\![\Gamma^{\geq t}]\!](\Delta^{\geq t})))$$

which follows from the induction hypothesis. ∎

We consider lists of constraints $L ::= R_1 s_1, \ldots, R_n s_n$ whose effect on a context is the evident cumulative filtering that extracts everything that satisfies every constraint in the list.

$$\Gamma^{R_1 s_1, \ldots, R_n s_n} = (\cdots(\Gamma^{R_1 s_1})\cdots)^{R_n s_n}$$

If P is a one-place predicate on signed worlds, we extend this predicate to constraint lists by requiring that every signed world in the constraint satisfies P, and to contexts in the same fashion.

$$P(R_1 s_1, \ldots, R_n s_n) \Leftrightarrow \forall i. P(s_i)$$
$$P(A_1[s_1], \ldots, A_n[s_n]) \Leftrightarrow \forall i. P(s_i)$$

Definition Let $\rightsquigarrow$ be a binary relation on propositions and P a one-place predicate on signed worlds. Then $\Gamma_1 \rightsquigarrow^P \Gamma_2$ is a binary relation between labelled contexts that is defined to hold iff $[\![\Gamma_1^L]\!](\Delta) \rightsquigarrow [\![\Gamma_2^L]\!](\Delta)$ for all L, Δ such that $P(L)$, $P(\Delta)$, and Δ is outside both Γ_1, Γ_2.

Lemma 1.5 *Suppose $\rightsquigarrow$ is a monotone preorder. If $\Gamma_1 \rightsquigarrow^P \Gamma_2$ and $P(\Gamma)$, then $\Gamma_1, \Gamma \rightsquigarrow^P \Gamma_2, \Gamma$.*

Proof By induction on Γ. The induction step is showing that if $\Gamma_1 \rightsquigarrow^P \Gamma_2$ and $P(s)$, then $\Gamma_1, A[s] \rightsquigarrow^P \Gamma_2, A[s]$. Let suitable L, Δ be given. We must show $[\![(\Gamma_1, A[s])^L]\!](\Delta) \rightsquigarrow [\![(\Gamma_2, A[s])^L]\!](\Delta)$. Let

$$P_{mn} = [\![\Gamma_m^{L,\not\geq s}]\!](\Delta^{\not\geq s}, A \odot^s ([\![\Gamma_n^{L,\geq s}]\!](\Delta^{\geq s})[s])$$

If $A[s]$ is erased by L, then we are done. Otherwise we need to show $P_{11} \rightsquigarrow P_{22}$. By induction hypothesis, $[\![\Gamma_1^{L,\geq s}]\!](\Delta^{\geq s}) \rightsquigarrow [\![\Gamma_2^{L,\geq s}]\!](\Delta^{\geq s})$ hence by monotonicity and reflexivity of $\rightsquigarrow$ we have $P_{11} \rightsquigarrow P_{12}$. By induction hypothesis again, $P_{12} \rightsquigarrow P_{22}$. By transitivity of $\rightsquigarrow$, we are done. ∎

Definition A binary relation $\rightsquigarrow$ on propositions is called *permutable* if

$$\bigvee_{i=1}^{n} A_i \rightsquigarrow \bigvee_{i=1}^{n} A_{\pi i}$$

for any permutation π

Lemma 1.6 *Suppose $\rightsquigarrow$ is a monotone, permutable binary relation. Then if Δ, Δ' are both outside Γ, and Δ' is a permutation of Δ, then $[\![\Gamma]\!](\Delta) \rightsquigarrow [\![\Gamma]\!](\Delta')$.*

Proof By induction on Γ. ■

Lemma 1.7 *Suppose $\rightsquigarrow$ is a monotone, permutable, swappable binary relation. Then $\Gamma, A_0[s_0], A_1[s_1] \rightsquigarrow^P \Gamma, A_1[s_1], A_0[s_0]$*

Proof Let suitable L, Δ be given. Assume without loss that L doesn't erase either $A_0[s_0]$ or $A_1[s_1]$. There are essentially three possible relationships between s_0 and s_1.

Case: $s_0 > s_1$, that is, $s_0 \geq s_1$ and $s_1 \not\geq s_0$. (The case where $s_1 > s_0$ is symmetric) Expanding out definitions on the left, (using only the fact that $s_0 \geq s_1$) we get

$$[\![\Gamma^{L,\not\geq s_1}]\!](\Delta^{\not\geq s_1}, A_1 \odot^{s_1} ([\![\Gamma^{L,\geq s_1}, A_0[s_0]]\!](\Delta^{\geq s_1}))[s_1]) =$$

$$[\![\Gamma^{L,\not\geq s_1,\not\geq s_0}]\!](\Delta^{\not\geq s_1,\not\geq s_0}, A_1 \odot^{s_1}$$
$$([\![\Gamma^{L,\geq s_1,\not\geq s_0}]\!](\Delta^{\geq s_1,\not\geq s_0}, A_0 \odot^{s_0}$$
$$([\![\Gamma^{L,\geq s_1,\geq s_0}]\!](\Delta^{\geq s_1,\geq s_0}))[s_0]))[s_1])$$

And expanding on the right (using both the known inequality and negated inequality) we get

$$[\![\Gamma^{L,\not\geq s_0}, A_1[s_1]]\!](\Delta^{\not\geq s_0}, A_0 \odot^{s_0} ([\![\Gamma^{L,\geq s_0}]\!](\Delta^{\geq s_0}))[s_0])$$

$$= [\![\Gamma^{L,\not\geq s_0,\not\geq s_1}]\!](\Delta^{\not\geq s_0,\not\geq s_1}, A_1 \odot^{s_1}$$
$$([\![\Gamma^{L,\not\geq s_0,\geq s_1}]\!](\Delta^{\not\geq s_0,\geq s_1}, A_0 \odot^{s_0}$$
$$([\![\Gamma^{L,\geq s_0,\geq s_1}]\!](\Delta^{\geq s_0,\geq s_1}))[s_0]))[s_1])$$

These expressions differ only in the order of application of constraints. Hence by reflexivity of $\rightsquigarrow$ (which follows from permutability) we are done.

Case: $s_0 = s_1$. Say both are equal to s. We can reuse our reasoning about the left-hand side from the previous case, and go further in this case because we know that $\Gamma^{\not\geq s_1,\not\geq s_0}$ is necessarily empty, and so too with $\Delta^{\not\geq s_1,\not\geq s_0}$. Hence it becomes

$$[\![\Gamma^{L,\not\geq s}]\!](\Delta^{\not\geq s}, A_1 \odot^{s} (A_0 \odot^{s} ([\![\Gamma^{L,\geq s}]\!](\Delta^{\geq s})))[s])$$

and the right-hand side is just

$$[\![\Gamma^{L,\not\geq s}]\!](\Delta^{\not\geq s}, A_0 \odot^s (A_1 \odot^s ([\![\Gamma^{L,\geq s}]\!](\Delta^{\geq s})))[s])$$

So by swappability and monotonicity they are $\rightsquigarrow$-related.

Case: $s_0 \perp s_1$, that is, $s_0 \not\geq s_1$ and $s_1 \not\geq s_0$. Here when we expand the left-hand side (taking advantage of the fact that if $s_0 \perp s_1$ and $s \geq s_0$, then $s \not\geq s_1$) we get

$$\begin{aligned}
&[\![\Gamma^{L,\not\geq s_1}, A_0[s_0]]\!](\Delta^{\not\geq s_1}, A_1 \odot^{s_1} ([\![\Gamma^{L,\geq s_1}]\!](\Delta^{\geq s_1}))[s_1]) \\
&\quad = [\![\Gamma^{L,\not\geq s_1,\not\geq s_0}]\!](\Delta^{\not\geq s_1,\not\geq s_0}, \\
&\quad\quad A_1 \odot^{s_1} ([\![\Gamma^{L,\geq s_1}]\!](\Delta^{\geq s_1}))[s_1], \\
&\quad\quad A_0 \odot^{s_0} ([\![\Gamma^{L,\geq s_0}]\!](\Delta^{\geq s_0}))[s_0])
\end{aligned}$$

and on the right-hand side we get the same thing with 0 and 1 interchanged. By Lemma 1.6 the one is $\rightsquigarrow$-related to the other.

■

Corollary 1.8 *Let $\rightsquigarrow$ be a monotone, permutable, swappable preorder. Suppose Γ' is a permutation of Γ. Then $\Gamma \rightsquigarrow^P \Gamma'$.*

Lemma 1.9 *Suppose $P(w) \Leftrightarrow w \not\geq pa$ and the binary relation $\rightsquigarrow$ is permutable. Then $A[pa^+] \rightsquigarrow^P A[p^+]$.*

Proof No constraint in P can discriminate p from pa, so we assume without loss that $A[pa^+]^L = A[pa^+]$ and $A[p^+]^L = A[p^+]$. We must establish that

$$[\![A[pa^+]]\!](\Delta) = v(\Delta) \vee A$$

is $\rightsquigarrow$-related to

$$[\![A[p^+]]\!](\Delta) = v(\Delta^{\not\geq p}) \vee (A \vee v(\Delta^{\geq p}))$$

but this follows immediately from permutability. ■

Lemma 1.10 *$\vdash$ and $\dashv\vdash$ are preorders that are*

- *monotone*
- *permutable*
- *swappable*
- *lex*

Furthermore, $\vdash$ is strongly swappable and weakenable.

Proof Transitivity follows from the cut elimination theorem, reflexivity from the hypothesis rule. Every other property is a simple proof in the sequent calculus. ■

Theorem 1.11 *If* $\Gamma \vdash_\ell \Delta$ *then* $\vdash [\![\Gamma^-, \Delta^+]\!](\cdot)$.

Proof By induction on the derivation of $\Gamma \vdash_\ell \Delta$. We abbreviate $\Omega = \Gamma^-, \Delta^+$ throughout.

Case:

$$\frac{}{\Gamma, A[p] \vdash_\ell A[pq], \Delta}$$

By Corollary 1.8 our goal might as well be showing that

$$\vdash [\![\Omega, A[p^-], A[pq^+]]\!](\cdot)$$

Expanding out the definitions this is

$$\vdash [\![\Omega^{\not\geq p}]\!](A \supset [\![\Omega^{\not\geq pq, \geq p}]\!](A \vee [\![\Omega^{\geq pq}]\!](\cdot)[pq])[p])$$

Using Lemma 1.4 and monotonicity we find

$$A \supset (A \vee X) \vdash A \supset [\![\Omega^{\not\geq pq, \geq p}]\!](A \vee X[pq])$$

for $X = [\![\Omega^{\geq pq}]\!](\cdot)$, and Lemma 1.4 once more on the outside and transitivity produces

$$A \supset (A \vee X) \vdash [\![\Omega^{\not\geq p}]\!](A \supset [\![\Omega^{\not\geq pq, \geq p}]\!](A \vee X[pq])[p])$$

but obviously we can cut away the $A \supset (A \vee X)$, and we are done.

Case:

$$\frac{\Gamma, A \supset B[p] \vdash_\ell A[pq], \Delta \qquad \Gamma, A \supset B[p], B[pq] \vdash_\ell \Delta}{\Gamma, A \supset B[p] \vdash_\ell \Delta}$$

By induction hypothesis we know

$$\vdash [\![\Gamma^-, A \supset B[p^-], A[pq^+], \Delta^+]\!](\cdot)$$

$$\vdash [\![\Gamma^-, A \supset B[p^-], B[pq^-], \Delta^+]\!](\cdot)$$

By Corollary 1.8 this is the same as knowing

$$\vdash [\![\Omega, A \supset B[p^-], A[pq^+]]\!](\cdot)$$

$$\vdash [\![\Omega, A \supset B[p^-], B[pq^-]]\!](\cdot)$$

Expanding out definitions we have

$$\vdash [\![\Omega^{\not\geq p}]\!]((A \supset B) \supset [\![\Omega^{\not\geq pq, \geq p}]\!](A \vee [\![\Omega^{\geq pq}]\!](\cdot)[pq])[p])$$

$$\vdash [\![\Omega^{\not\geq p}]\!]((A \supset B) \supset [\![\Omega^{\not\geq pq, \geq p}]\!](B \supset [\![\Omega^{\geq pq}]\!](\cdot)[pq])[p])$$

Using montonicity and lexity we can obtain from these that

$$\vdash [\![\Omega^{\not\geq p}]\!]((A \supset B) \supset [\![\Omega^{\not\geq pq, \geq p}]\!]((A \vee X) \wedge (B \supset X)[pq])[p]) \qquad (*)$$

for $X = [\![\Omega^{\geq pq}]\!](\cdot)[pq]$. It is easy to show that

$$(A \vee X) \wedge (B \supset X) \vdash (A \supset B) \supset X$$

so using monotonicity and strong swappability, writing $F(X)$ in place of $[\![\Omega^{\not\geq pq, \geq p}]\!](X[pq])$, we can see

$$F((A \vee X) \wedge (B \supset X)) \vdash F((A \supset B) \supset X) \vdash (A \supset B) \supset F(X)$$

hence monotonicity and fact that $C \supset (C \supset D) \vdash C \supset D$ for any C, D give us

$$(A \supset B) \supset F((A \vee X) \wedge (B \supset X)) \vdash$$
$$(A \supset B) \supset (A \supset B) \supset F(X) \vdash (A \supset B) \supset F(X)$$

hence by monotonicity one final time we find

$$[\![\Omega^{\not\geq p}]\!]((A \supset B) \supset F((A \vee X) \wedge (B \supset X))[p]) \vdash [\![\Omega^{\not\geq p}]\!]((A \supset B) \supset F(X)[p])$$

which can be cut against $(*)$ to produce the required result.

Case:

$$\frac{\Gamma, A[pa] \vdash_\ell B[pa], \Delta}{\Gamma \vdash_\ell A \supset B[p], \Delta}$$

By induction hypothesis we know

$$\vdash [\![\Gamma^-, A[pa^-], B[pa^+], \Delta^+]\!](\cdot)$$

and we must show

$$\vdash [\![\Gamma^-, A \supset B[p^+], \Delta^+]\!](\cdot)$$

By Corollary 1.8 this amounts to having

$$\vdash [\![A[pa^-], B[pa^+], \Omega]\!](\cdot)$$

and needing to show

$$\vdash [\![A \supset B[p^+], \Omega]\!](\cdot)$$

By Lemma 1.9 we know that

$$A[pa^-], B[pa^+] \vdash^P A \supset B[p^+]$$

for $P(w) \Leftrightarrow w \not\geq pa$. Now $P(\Omega)$, because the world label a is fresh. So Lemma 1.5 implies that

$$A[pa^-], B[pa^+], \Omega \vdash^P A \supset B[p^+], \Omega$$

from which we can extract the special case of

$$[\![A[pa^-], B[pa^+], \Omega]\!](\cdot) \vdash [\![A \supset B[p^+], \Omega]\!](\cdot)$$

and we are done.

■