$$\frac{\Gamma, A \supset B[p] \vdash_\ell A[pq], \Delta \qquad \Gamma, A \supset B[p], B[pq] \vdash_\ell \Delta}{\Gamma, A \supset B[p] \vdash_\ell \Delta}$$

$$\frac{\Gamma, A[pa] \vdash_\ell B[pa], \Delta}{\Gamma \vdash_\ell A \supset B[p], \Delta}$$

$$\frac{}{\Gamma, A[p] \vdash_\ell A[pq], \Delta}$$

$$\frac{\Gamma \vdash_\ell A[p], B[p], \Delta}{\Gamma \vdash_\ell A \vee B[p], \Delta}$$

$$\frac{\Gamma, A[p] \vdash_\ell \Delta \qquad \Gamma, B[p] \vdash \Delta}{\Gamma, A \vee B[p] \vdash_\ell \Delta}$$

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1 Soundness of Labelled Sequent Calculus

The basic inequality $\geq$ is of course defined by saying $p \geq r$ if r is of the form pq. We have a notion of *signed worlds* p^+ or p^- (for which we use the variables s, t) that have an annotation indicating whether they came from a positive or negative position in the context. Essentially we work with a one-sided sequent. The inequality $\geq$ is extended to signed worlds by saying

$$p^i \geq r^j \Leftrightarrow p \geq r \vee (p = r \wedge i = + \wedge j = -)$$

For $R \in \geq, \not\geq$ and any signed world s, the expression Rs is a *constraint.* We indicate a context filtered out by a constraint by writing the latter as a superscript, e.g. $\Gamma^{\geq s}$ is that part of Γ in the future of world s.

$$(\Gamma, A[t])^{Rs} = \begin{cases} \Gamma^{Rs}, A^i[t] & \text{if } tRs; \\ \Gamma^{Rs} & \text{otherwise.} \end{cases}$$

$$(\cdot)^{Rs} = \cdot$$

The central definition of the function $[\![\Gamma]\!](\Delta)$ — which realizes sequents as propositions — is as follows:

$$A \odot^{w^+} B = \begin{cases} A & \text{if } B = \bot; \\ A \vee B & \text{otherwise.} \end{cases}$$

$$A \odot^{w^-} B = A \supset B$$

$$[\![\Gamma, A[s]]\!](\Delta) = [\![\Gamma^{\not\geq s}]\!](\Delta^{\not\geq s}, A \odot^s ([\![\Gamma^{\geq s}]\!](\Delta^{\geq s}))[s])$$

$$[\![\cdot]\!](\Delta) = v(\Delta)$$

$$v(A_1[s_1], \ldots, A_n[s_n]) = \bigvee_{i=1}^{n} A_n$$

We also consider list of constraints $C ::= R_1 s_1, \ldots, R_n s_n$ whose effect on a context is the evident cumulative filtering that extracts everything that satisfies every constraint in the list.

$$\Gamma^{R_1 s_1, \ldots, R_n s_n} = (\cdots (\Gamma^{R_1 s_1}) \cdots)^{R_n s_n}$$

If P is a one-place predicate on signed worlds, we extend this predicate to constraint lists by requiring that every signed world in the constraint satisfies P, and to contexts in the same fashion.

$$P(R_1 s_1, \ldots, R_n s_n) \Leftrightarrow \forall i. P(s_i)$$

$$P(A_1[s_1], \ldots, A_n[s_n]) \Leftrightarrow \forall i. P(s_i)$$

Definition Δ is said to be *outside* Γ if $s \not\geq t$ for every $A[s] \in \Gamma, B[t] \in \Delta$.

Definition Let $\sqsubset$ be a binary relation on propositions and P a one-place predicate on signed worlds. Then $\Gamma_1 \sqsubset^P \Gamma_2$ is a binary relation between labelled contexts that is defined to hold iff $[\![\Gamma_1^C]\!](\Delta) \sqsubset [\![\Gamma_2^C]\!](\Delta)$ for all C, Δ such that $P(C)$, $P(\Delta)$, and Δ is outside both Γ_1, Γ_2.

Definition A binary relation $\sqsubset$ on propositions is called *monotone* if

- $A \odot^s B \sqsubset A \odot B'$ for any s and any $B \sqsubset B'$
- $[\![\Gamma]\!](A_1^1[s_1], \ldots, A_n^1[s_n]) \sqsubset [\![\Gamma]\!](A_1^2[s_1], \ldots, A_n^2[s_n])$ for any Γ such that the context $(A_1^k[s_1], \ldots, A_n^k[s_n])$ is outside Γ for each $k \in \{1, 2\}$, and $A_m^1 \sqsubset A_m^2$ for all $m \in 1 \ldots n$.

Definition A binary relation $\sqsubset$ on propositions is called *swappable* if

$$A \odot^s (B \odot^s C) \sqsubset B \odot^s (A \odot^s C)$$

for any A, B, C, s.

Definition A binary relation $\sqsubset$ on propositions is called *permutable* if

$$\bigvee_{i=1}^{n} A_i \sqsubset \bigvee_{i=1}^{n} A_{\pi i}$$

for any permutation π

Lemma 1.1 *Suppose $\sqsubset$ is a monotone preorder. If $\Gamma_1 \sqsubset^P \Gamma_2$ and $P(\Gamma)$, then $\Gamma_1, \Gamma \sqsubset^P \Gamma_2, \Gamma$.*

Proof By induction on Γ. The induction step is showing that if $\Gamma_1 \sqsubset^P \Gamma_2$ and $P(s)$, then $\Gamma_1, A[s] \sqsubset^P \Gamma_2, A[s]$. Let suitable C, Δ be given. We must show $[\![(\Gamma_1, A[s])^C]\!](\Delta) \sqsubset [\![(\Gamma_2, A[s])^C]\!](\Delta)$. Let

$$P_{mn} = [\![\Gamma_m^{C, \not\geq s}]\!](\Delta^{\not\geq s}, A \odot^s ([\![\Gamma_n^{C, \geq s}]\!](\Delta^{\geq s})[s])$$

If $A[s]$ is erased by C, then we are done. Otherwise we need to show $P_{11} \sqsubset P_{22}$. By induction hypothesis, $[\![\Gamma_1^{C, \geq s}]\!](\Delta^{\geq s}) \sqsubset [\![\Gamma_2^{C, \geq s}]\!](\Delta^{\geq s})$ hence by monotonicity and reflexivity of $\sqsubset$ we have $P_{11} \sqsubset P_{12}$. By induction hypothesis again, $P_{12} \sqsubset P_{22}$. By transitivity of $\sqsubset$, we are done. ∎

Lemma 1.2 *Suppose $\sqsubset$ is a monotone, permutable binary relation. Then if Δ, Δ' are both outside Γ, and Δ' is a permutation of Δ, then $[\![\Gamma]\!](\Delta) \sqsubset [\![\Gamma]\!](\Delta')$.*

Proof By induction. ∎

Lemma 1.3 *Suppose $\sqsubset$ is a monotone, permutable, swappable binary relation. Then $\Gamma, A_0[s_0], A_1[s_1] \sqsubset^P \Gamma, A_1[s_1], A_0[s_0]$*

Proof Let suitable C, Δ be given. Assume without loss that C doesn't erase either $A_0[s_0]$ or $A_1[s_1]$. There are essentially three possible relationships between s_0 and s_1.

Case: $s_0 > s_1$, that is, $s_0 \geq s_1$ and $s_1 \not\geq s_0$. (The case where $s_1 > s_0$ is symmetric) Expanding out definitions on the left, (using only the fact that $s_0 \geq s_1$) we get

$$[\![\Gamma^{C,\not\geq s_1}]\!](\Delta^{\not\geq s_1}, A_1 \odot^{s_1} ([\![\Gamma^{C,\geq s_1}, A_0[s_0]]\!](\Delta^{\geq s_1}))[s_1]) =$$

$$[\![\Gamma^{C,\not\geq s_1,\not\geq s_0}]\!](\Delta^{\not\geq s_1,\not\geq s_0}, A_1\odot^{s_1}$$
$$([\![\Gamma^{C,\geq s_1,\not\geq s_0}]\!](\Delta^{\geq s_1,\not\geq s_0}, A_0\odot^{s_0}$$
$$([\![\Gamma^{C,\geq s_1,\geq s_0}]\!](\Delta^{\geq s_1,\geq s_0}))[s_0]))[s_1])$$

And expanding on the right (using both the known inequality and negated inequality) we get

$$[\![\Gamma^{C,\not\geq s_0}, A_1[s_1]]\!](\Delta^{\not\geq s_0}, A_0 \odot^{s_0} ([\![\Gamma^{C,\geq s_0}]\!](\Delta^{\geq s_0}))[s_0])$$

$$= [\![\Gamma^{C,\not\geq s_0,\not\geq s_1}]\!](\Delta^{\not\geq s_0,\not\geq s_1}, A_1\odot^{s_1}$$
$$([\![\Gamma^{C,\not\geq s_0,\geq s_1}]\!](\Delta^{\not\geq s_0,\geq s_1}, A_0\odot^{s_0}$$
$$([\![\Gamma^{C,\geq s_0,\geq s_1}]\!](\Delta^{\geq s_0,\geq s_1}))[s_0]))[s_1])$$

These expressions differ only in the order of application of constraints. Hence by reflexivity of $\sqsubset$ (which follows from permutability) we are done.

Case: $s_0 = s_1$. Say both are equal to s. We can reuse our reasoning about the left-hand side from the previous case, and go further in this case because we know that $\Gamma^{\not\geq s_1,\not\geq s_0}$ is necessarily empty, and so too with $\Delta^{\not\geq s_1,\not\geq s_0}$. Hence it becomes

$$[\![\Gamma^{C,\not\geq s}]\!](\Delta^{\not\geq s}, A_1 \odot^s (A_0 \odot^s ([\![\Gamma^{C,\geq s}]\!](\Delta^{\geq s})))[s])$$

and the right-hand side is just

$$[\![\Gamma^{C,\not\geq s}]\!](\Delta^{\not\geq s}, A_0 \odot^s (A_1 \odot^s ([\![\Gamma^{C,\geq s}]\!](\Delta^{\geq s})))[s])$$

So by swappability and monotonicity they are $\sqsubset$-related.

Case: $s_0 \perp s_1$, that is, $s_0 \not\geq s_1$ and $s_1 \not\geq s_0$. Here when we expand the left-hand side (taking advantage of the fact that if $s_0 \perp s_1$ and $s \geq s_0$, then $s \not\geq s_1$) we get

$$\llbracket \Gamma^{C,\not\geq s_1}, A_0[s_0] \rrbracket (\Delta^{\not\geq s_1}, A_1 \odot^{s_1} (\llbracket \Gamma^{C,\geq s_1} \rrbracket (\Delta^{\geq s_1}))[s_1])$$

$$\begin{aligned} &= \llbracket \Gamma^{C,\not\geq s_1,\not\geq s_0} \rrbracket (\Delta^{\not\geq s_1,\not\geq s_0}, \\ &\quad A_1 \odot^{s_1} (\llbracket \Gamma^{C,\geq s_1} \rrbracket (\Delta^{\geq s_1}))[s_1], \\ &\quad A_0 \odot^{s_0} (\llbracket \Gamma^{C,\geq s_0} \rrbracket (\Delta^{\geq s_0}))[s_0]) \end{aligned}$$

and on the right-hand side we get the same thing with 0 and 1 interchanged. By Lemma 1.2 the one is $\sqsubset$-related to the other.

■

Corollary 1.4 *Let $\sqsubset$ be a monotone, permutable, swappable preorder. Suppose Γ' is a permutation of Γ. Then $\Gamma \sqsubset^P \Gamma'$.*

Lemma 1.5 *$\vdash$ and $\dashv\vdash$ are monotone, permutable, swappable preorders.*

Proof Transitivity follows from the cut elimination theorem. The first clause of monotonicity can be proved directly: If $B \dashv\vdash B'$, then $A \supset B \dashv\vdash A \supset B'$ and $A \vee B \dashv\vdash A \vee B'$ and if $B \dashv\vdash \bot$, then $A \vee B \dashv\vdash A$, etc. The second clause of monotonicity is satisfied by an easy inductive argument. Permutability and swappability are direct simple proofs in the sequent calculus. ■

Lemma 1.6 *Suppose $P(w) \Leftrightarrow w \not\geq pa$ and the binary relation $\sqsubset$ is permutable. Then $A[pa^+] \sqsubset^P A[p^+]$.*

Proof No constraint in P can discriminate p from pa, so we assume without loss that $A[pa^+]^C = A[pa^+]$ and $A[p^+]^C = A[p^+]$. We must establish that

$$\llbracket A[pa^+] \rrbracket (\Delta) = v(\Delta) \vee A$$

is $\sqsubset$-related to

$$\llbracket A[p^+] \rrbracket (\Delta) = v(\Delta^{\not\geq p}) \vee A \vee v(\Delta^{\geq p})$$

but this follows immediately from permutability. ■

The thing we aim to prove is

Proposition 1.7 *If $\Gamma \vdash_\ell \Delta$ then $\vdash \llbracket \Gamma^-, \Delta^+ \rrbracket (\cdot)$.*