The Complexity of Computing Minimal Unidirectional Covering Sets

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Abstract

A common thread in the social sciences is to identify sets of alternatives that satisfy certain notions of stability according to some binary dominance relation. Examples can be found in areas as diverse as voting theory, game theory, and argumentation theory. Brandt and Fischer \cite{BrandtFischer08} proved that it is NP-hard to decide whether an alternative is contained in some inclusion-minimal unidirectional (i.e., either upward or downward) covering set. For both problems, we raise this lower bound to the $\Theta_2^p$ level of the polynomial hierarchy and provide a $\Sigma_2^p$ upper bound. Relatedly, we show that a variety of other natural problems regarding minimal or minimum-size unidirectional covering sets are hard or complete for either of NP, coNP, and $\Theta_2^p$. An important consequence of our results is that neither minimal upward nor minimal downward covering sets (even when guaranteed to exist) can be computed in polynomial time unless P = NP. This sharply contrasts with Brandt and Fischer’s result that minimal bidirectional covering sets are polynomial-time computable.

1. Introduction

A common thread in the social sciences is to identify sets of alternatives that satisfy certain notions of stability according to some binary dominance relation. Applications range from cooperative to non-cooperative game theory, from social choice theory to argumentation theory, and from multi-criteria decision analysis to sports tournaments (see, e.g., \cite{BaumeisterBrandtFischerHoffmann10,BrandtFischer08} and the references therein).

In settings of social choice, the most common dominance relation is the pairwise majority relation, where an alternative $x$ is said to dominate another alternative $y$ if the number of individuals preferring $x$ to $y$ exceeds the number of individuals preferring $y$ to $x$. McGarvey \cite{McGarvey94} proved that every asymmetric dominance relation can be realized via a particular preference

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profile, even if the individual preferences are linear. The dominance graph shown in Figure 1 may for example result from the individual preferences of six voters given in the following table where each column represents a number of voters with preferences given in decreasing order. For example, the first column represents two voters who rank the alternatives in alphabetical order.

<table>
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A well-known paradox due to the Marquis de Condorcet [5] says that the majority relation may contain cycles and thus does not always admit maximal elements, even if all of the underlying individual preferences do. This means that the concept of maximality is rendered useless in most cases. For this reason, various alternative solution concepts that can be used in place of maximality for nontransitive relations (see, e.g., [3]) have been proposed. In particular, concepts based on covering relations—transitive subrelations of the dominance relation at hand—have turned out to be very attractive [6, 7, 8].

In this paper, we study the computational complexity of problems related to the notions of upward and downward covering sets in dominance graphs. An alternative $x$ is said to upward cover another alternative $y$ if $x$ dominates $y$ and every alternative dominating $x$ also dominates $y$. The intuition is that $x$ “strongly” dominates $y$ in the sense that there is no alternative that dominates $x$ but not $y$. Similarly, an alternative $x$ is said to downward cover another alternative $y$ if $x$ dominates $y$ and every alternative dominated by $y$ is also dominated by $x$. The intuition here is that $x$ “strongly” dominates $y$ in the sense that there is no alternative dominated by $y$ but not by $x$. A minimal upward or minimal downward covering set is defined as an inclusion-minimal set of alternatives that satisfies certain notions of internal and external stability with respect to the upward or downward covering relation [8, 1].

Recent work in computational social choice has addressed the computational complexity of most solution concepts proposed in the context of binary dominance (see, e.g., [9, 10, 11, 12, 1, 13]). In particular, Brandt and Fischer [1] have shown NP-hardness of both the problem of deciding whether an alternative is contained in some minimal upward covering set and the problem of deciding whether an alternative is contained in some minimal downward covering set. For both problems, we improve on these results by raising their NP-hardness lower bounds to the $\Theta_2^p$ level of the polynomial hierarchy, and we provide an upper bound of $\Sigma_2^p$. Moreover, we will analyze the complexity of a variety of other problems related to minimal and minimum-size upward and downward covering sets that have not been studied before. In particular, we provide hardness and completeness results for the complexity classes NP, coNP, and $\Theta_2^p$. Remarkably, these new results imply that neither minimal upward covering sets nor minimal downward covering sets (even when guaranteed to exist) can be found in polynomial time unless $P = NP$. This sharply contrasts with Brandt and Fischer’s result that minimal bidirectional covering sets are polynomial-time computable [1]. Note that, notwithstanding the hardness of computing minimal upward covering sets, the decision version of this
search problem is trivially in P: Every dominance graph always contains a minimal upward covering set.

Our $\Theta^P_2$-hardness results apply Wagner’s method [14] that was useful also in other contexts (see, e.g., [14, 15, 16, 17, 18]). To the best of our knowledge, our constructions for the first time apply his method to problems defined in terms of minimality rather than minimum size of a solution.

2. Definitions and Notation

In this section, we define the necessary concepts from social choice theory and complexity theory.

Definition 2.1 (Covering Relations). Let $A$ be a finite set of alternatives, let $B \subseteq A$, and let $\succ \subseteq A \times A$ be a dominance relation on $A$, i.e., $\succ$ is asymmetric and irreflexive.\(^1\) A dominance relation $\succ$ on a set $A$ of alternatives can be conveniently represented as a dominance graph, denoted by $(A, \succ)$, whose vertices are the alternatives from $A$, and for each $x, y \in A$ there is a directed edge from $x$ to $y$ if and only if $x \succ y$.

For any two alternatives $x$ and $y$ in $B$, define the following covering relations (see, e.g., [6, 7, 19]):

- $x$ upward covers $y$ in $B$, denoted by $x \mathcal{C}_u y$, if $x \succ y$ and for all $z \in B$, $z \succ x$ implies $z \succ y$, and
- $x$ downward covers $y$ in $B$, denoted by $x \mathcal{C}_d y$, if $x \succ y$ and for all $z \in B$, $y \succ z$ implies $x \succ z$.

When clear from the context, we omit mentioning “in $B$” explicitly and simply write $xC_u y$ rather than $xC_u^B y$, and $xC_d y$ rather than $xC_d^B y$.

Definition 2.2 (Uncovered Set). Let $A$ be a set of alternatives, let $B \subseteq A$ be any subset, let $\succ$ be a dominance relation on $A$, and let $C$ be a covering relation on $A$ based on $\succ$. The uncovered set of $B$ with respect to $C$ is defined as

$$UC_C(B) = \{ x \in B \mid y C x \text{ for no } y \in B \}.$$  

For notational convenience, let $UC_x(B) = UC_{C_x}(B)$ for $x \in \{u, d\}$, and we call $UC_u(B)$ the upward uncovered set of $B$ and $UC_d(B)$ the downward uncovered set of $B$.

In the dominance graph $(A, \succ)$ in Figure 1, $b$ upward covers $c$ in $A$, and $a$ downward covers $b$ in $A$ (i.e., $b \mathcal{C}_u^A c$ and $a \mathcal{C}_d^A b$), so $UC_u(A) = \{a, b, d\}$ is the upward uncovered set and $UC_d(A) = \{a, c, d\}$ is the downward uncovered set of $A$. For both the upward and the downward covering relation (henceforth both will be called unidirectional covering relations),

\(^1\)In general, $\succ$ need not be transitive or complete. For alternatives $x$ and $y$, $x \succ y$ (equivalently, $(x, y) \in \succ$) is interpreted as $x$ being strictly preferred to $y$ (and we say “$x$ dominates $y$”), e.g., due to a strict majority of voters preferring $x$ to $y$. 

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transitivity of the relation implies nonemptiness of the corresponding uncovered set for each nonempty set of alternatives. The intuition underlying covering sets is that there should be no reason to restrict the selection by excluding some alternative from it (internal stability) and there should be an argument against each proposal to include an outside alternative into the selection (external stability).

Definition 2.3 (Minimal Covering Set). Let $A$ be a set of alternatives, let $\succ$ be a dominance relation on $A$, and let $C$ be a covering relation based on $\succ$. A subset $B \subseteq A$ is a covering set for $A$ under $C$ if the following two properties hold:

- Internal stability: $UC_C(B) = B$.
- External stability: For all $x \in A - B$, $x \not\in UC_C(B \cup \{x\})$.

A covering set $M$ for $A$ under $C$ is said to be (inclusion-)minimal if no $M' \subset M$ is a covering set for $A$ under $C$.

Every upward uncovered set contains one or more minimal upward covering sets, whereas minimal downward covering sets may not always exist [1]. Dutta [8] proposed minimal covering sets in the context of tournaments, i.e., complete dominance relations. In tournaments, both notions of covering coincide because the set of alternatives dominating a given alternative $x$ consists precisely of those alternatives not dominated by $x$. Minimal unidirectional covering sets are one of several possible generalizations to incomplete dominance relations (for more details, see [1]). Occasionally, it might be helpful to specify the dominance relation explicitly to avoid ambiguity. In such cases we refer to the dominance graph used and write, e.g., “$M$ is an upward covering set for $(A, \succ)$.” The unique minimal upward covering set for the dominance graph shown in Figure 1 is $\{b, d\}$, and the unique minimal downward covering set is $\{a, c, d\}$.

In addition to the (inclusion-)minimal unidirectional covering sets considered by Brandt and Fischer [1], we also consider minimum-size covering sets, i.e., unidirectional covering sets of smallest cardinality. For some of the computational problems we study, different complexities can be shown for the minimal and minimum-size versions of the problem (see Theorem 3.1 and Table 1). Specifically, we consider six types of computational problems, for both upward and downward covering sets, and for each both their “minimal” (prefixed by $MC_u$ or $MC_d$) and “minimum-size” (prefixed by $MSC_u$ or $MSC_d$) versions. We first define the six problem types for the case of minimal upward covering sets:

1. $MC_u$-Size: Given a set $A$ of alternatives, a dominance relation $\succ$ on $A$, and a positive integer $k$, does there exist some minimal upward covering set for $A$ containing at most $k$ alternatives?

Figure 1: Dominance graph $(A, \succ)$. 
2. **MCₚ-Member**: Given a set \( A \) of alternatives, a dominance relation \( \succ \) on \( A \), and a distinguished element \( d \in A \), is \( d \) contained in some minimal upward covering set for \( A \)?

3. **MCₚ-Member-All**: Given a set \( A \) of alternatives, a dominance relation \( \succ \) on \( A \), and a distinguished element \( d \in A \), is \( d \) contained in all minimal upward covering sets for \( A \)?

4. **MCₚ-Unique**: Given a set \( A \) of alternatives and a dominance relation \( \succ \) on \( A \), does there exist a unique minimal upward covering set for \( A \)?

5. **MCₚ-Test**: Given a set \( A \) of alternatives, a dominance relation \( \succ \) on \( A \), and a subset \( M \subseteq A \), is \( M \) a minimal upward covering set for \( A \)?

6. **MCₚ-Find**: Given a set \( A \) of alternatives and a dominance relation \( \succ \) on \( A \), find a minimal upward covering set for \( A \).

If we replace “upward” by “downward” above, we obtain the six corresponding “downward covering” versions, denoted by \( \text{MC}_{\text{d}} \text{-Size} \), \( \text{MC}_{\text{d}} \text{-Member} \), \( \text{MC}_{\text{d}} \text{-Member-All} \), \( \text{MC}_{\text{d}} \text{-Unique} \), \( \text{MC}_{\text{d}} \text{-Test} \), and \( \text{MC}_{\text{d}} \text{-Find} \). And if we replace “minimal” by “minimum-size” in the twelve problems just defined, we obtain the corresponding “minimum-size” versions: \( \text{MSC}_{\text{u}} \text{-Size} \), \( \text{MSC}_{\text{u}} \text{-Member} \), \( \text{MSC}_{\text{u}} \text{-Member-All} \), \( \text{MSC}_{\text{u}} \text{-Unique} \), \( \text{MSC}_{\text{u}} \text{-Test} \), \( \text{MSC}_{\text{u}} \text{-Find} \), \( \text{MSC}_{\text{d}} \text{-Size} \), \( \text{MSC}_{\text{d}} \text{-Member} \), \( \text{MSC}_{\text{d}} \text{-Member-All} \), \( \text{MSC}_{\text{d}} \text{-Unique} \), \( \text{MSC}_{\text{d}} \text{-Test} \), and \( \text{MSC}_{\text{d}} \text{-Find} \).

Note that the four problems \( \text{MC}_{\text{u}} \text{-Find} \), \( \text{MC}_{\text{d}} \text{-Find} \), \( \text{MSC}_{\text{u}} \text{-Find} \), and \( \text{MSC}_{\text{d}} \text{-Find} \) are search problems, whereas the other twenty problems are decision problems.

We assume that the reader is familiar with the basic notions of complexity theory, such as polynomial-time many-one reducibility and the related notions of hardness and completeness, and also with standard complexity classes such as P, NP, coNP, and the polynomial hierarchy [20] (see also, e.g., the textbooks [21, 22]). In particular, coNP is the class of sets whose complements are in NP. \( \Sigma^p_2 = \text{NP}^{\text{NP}} \), the second level of the polynomial hierarchy, consists of all sets that can be solved by an NP oracle machine that has access (in the sense of a Turing reduction) to an NP oracle set such as SAT. SAT denotes the satisfiability problem of propositional logic, which is one of the standard NP-complete problems (see, e.g., Garey and Johnson [23]) and is defined as follows: Given a boolean formula in conjunctive normal form, does there exist a truth assignment to its variables that satisfies the formula?

Papadimitriou and Zachos [24] introduced the class of problems solvable in polynomial time via asking \( O(\log n) \) sequential Turing queries to NP. This class is also known as the \( \Theta^p_2 \) level of the polynomial hierarchy (see Wagner [25]), and has been shown to coincide with the class of problems that can be decided by a P machine that accesses its NP oracle in a parallel manner (see [26, 27]). Equivalently, \( \Theta^p_2 \) is the closure of NP under polynomial-time truth-table reductions. It follows immediately from the definitions that \( P \subseteq \text{NP} \cap \text{coNP} \subseteq \text{NP} \cup \text{coNP} \subseteq \Theta^p_2 \subseteq \Sigma^p_2 \).

\( \Theta^p_2 \) captures the complexity of various optimization problems. For example, the problem of testing whether the size of a maximum clique in a given graph is an odd number, the problem of deciding whether two given graphs have minimum vertex covers of the same size, and the problem of recognizing those graphs for which certain heuristics yield good approximations for the size of a maximum independent set or for the size of a minimum vertex cover each are
Table 1: Overview of complexity results for the various types of covering set problems. As indicated, previously known results are due to Brandt and Fischer [1]; all other results are new to this paper.

<table>
<thead>
<tr>
<th>Problem Type</th>
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<th>MSC_u</th>
<th>MC_d</th>
<th>MSC_d</th>
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<td>NP-complete</td>
<td>NP-complete</td>
<td>NP-complete</td>
</tr>
<tr>
<td>Member</td>
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<td>Θ^p_2-complete</td>
<td>Θ^p_2-hard and in Σ^p_2</td>
<td>coNP-hard and in Θ^p_2</td>
</tr>
<tr>
<td>Member-All</td>
<td>coNP-complete [1]</td>
<td>Θ^p_2-complete</td>
<td>coNP-complete [1]</td>
<td>coNP-hard and in Θ^p_2</td>
</tr>
<tr>
<td>Unique</td>
<td>coNP-hard and in Σ^p_2</td>
<td>coNP-hard and in Θ^p_2</td>
<td>coNP-hard and in Σ^p_2</td>
<td>coNP-hard and in Θ^p_2</td>
</tr>
<tr>
<td>Test</td>
<td>coNP-complete</td>
<td>coNP-complete</td>
<td>coNP-complete</td>
<td>coNP-complete</td>
</tr>
<tr>
<td>Find</td>
<td>not in polynomial</td>
<td>not in polynomial</td>
<td>not in polynomial</td>
<td>time unless P = NP</td>
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</table>

Table 1: Overview of complexity results for the various types of covering set problems. As indicated, previously known results are due to Brandt and Fischer [1]; all other results are new to this paper.

known to be complete for Θ^p_2 (see [14, 16, 18]). Hemaspaandra and Wechsung [17] proved that the minimization problem for boolean formulas is Θ^p_2-hard. In the field of computational social choice, the winner problems for Dodgson [28], Young [29], and Kemeny [30] elections have been shown to be Θ^p_2-complete in the nonunique-winner model [15, 31, 32], and also in the unique-winner model [33].

3. Results and Discussion

Results. Brandt and Fischer [1] proved that it is NP-hard to decide whether a given alternative is contained in some minimal unidirectional covering set. Using the notation of this paper, their results state that the problems MC_u-Member and MC_d-Member are NP-hard. The question of whether these two problems are NP-complete or of higher complexity was left open in [1]. Our contribution is

1. to raise Brandt and Fischer’s NP-hardness lower bounds for MC_u-Member and MC_d-Member to Θ^p_2-hardness and to provide (simple) Σ^p_2 upper bounds for these problems, and

2. to extend the techniques we developed to apply also to the 22 other covering set problems defined in Section 2, in particular to the search problems.

Our results are stated in the following theorem.

Theorem 3.1. The complexity of the covering set problems defined in Section 2 is as shown in Table 1.

The detailed proofs of the single results collected in Theorem 3.1 will be presented in Section 5, and the technical constructions establishing the properties that are needed for these proofs are given in Section 4.
Discussion. We consider the problems of finding minimal and minimum-size upward and downward covering sets (MC$_u$-Find, MC$_d$-Find, MSC$_u$-Find, and MSC$_d$-Find) to be particularly important and natural.

Regarding upward covering sets, we stress that our result (see Theorem 5.7) that, assuming $P \neq NP$, MC$_u$-Find and MSC$_u$-Find are hard to compute does not seem to follow directly from the NP-hardness of MC$_u$-Member in any obvious way. The decision version of MC$_u$-Find is: Given a dominance graph, does it contain a minimal upward covering set? However, this question has always an affirmative answer, so the decision version of MC$_u$-Find is trivially in $P$. Note also that MC$_u$-Find can be reduced in a “disjunctive truth-table” fashion to the search version of MC$_u$-Member (“Given a dominance graph $(A, \succ)$ and an alternative $d \in A$, find some minimal upward covering set for $A$ that contains $d$”) by asking this oracle set about all alternatives in parallel. So MC$_u$-Find is no harder (with respect to disjunctive truth-table reductions) than that problem. The converse, however, is not at all obvious. Brandt and Fischer’s results only imply the hardness of finding an alternative that is contained in all minimal upward covering sets [1]. Our reduction that raises the lower bound of MC$_u$-Member from NP-hardness to $\Theta^p_2$-hardness, however, also allows us to prove that MC$_u$-Find and MSC$_u$-Find cannot be solved in polynomial time unless $P = NP$.

Regarding downward covering sets, the result that MC$_d$-Find cannot be computed in polynomial time unless $P = NP$ is an immediate consequence of Brandt and Fischer’s result that it is NP-complete to decide whether there exists a minimal downward covering set [1, Thm. 9]. We provide an alternative proof based on our reduction showing that MC$_d$-Member is $\Theta^p_2$-hard (see the proof of Theorem 5.13). In contrast to Brandt and Fischer’s proof, our proof shows that MC$_d$-Find is hard to compute even when the existence of a (minimal) downward covering set is guaranteed. As indicated in Table 1, coNP-completeness of MC$_u$-Member-All and MC$_d$-Member-All was also shown previously by Brandt and Fischer [1].

As mentioned above, the two problems MC$_u$-Member and MC$_d$-Member were already known to be NP-hard [1] and are here shown to be even $\Theta^p_2$-hard. One may naturally wonder whether raising their (or any problem’s) lower bound from NP-hardness to $\Theta^p_2$-hardness gives us any more insight into the problem’s inherent computational complexity. After all, $P = NP$ if and only if $P = \Theta^p_2$. However, this question is a bit more subtle than that and has been discussed carefully by Hemaspaandra et al. [34]. They make the case that the answer to this question crucially depends on what one considers to be the most natural computational model. In particular, they argue that raising NP-hardness to $\Theta^p_2$-hardness potentially (i.e., unless longstanding open problems regarding the separation of the corresponding complexity classes could be solved) is an improvement in terms of randomized polynomial time and in terms of unambiguous polynomial time [34].

4. Constructions

In this section, we provide the constructions that will be used in Section 5 to obtain the new complexity results for the problems defined in Section 2.

4.1. Minimal and Minimum-Size Upward Covering Sets

We start by giving the constructions that will be used for establishing results on the minimal and minimum-size upward covering set problems. Brandt and Fischer [1] proved the
following result. Since we need their reduction in Construction 4.7 and Section 5, we give a
proof sketch for Theorem 4.1.

Theorem 4.1 (Brandt and Fischer [1]). Deciding whether a designated alternative is con-
tained in some minimal upward covering set for a given dominance graph is
\(\text{NP-hard}\). That is, \(\text{MC}_u\text{-MEMBER}\) is \(\text{NP-hard}\).

Proof Sketch. NP-hardness is shown by a reduction from SAT. Given a boolean formula in
conjunctive normal form, \(\varphi(v_1, v_2, \ldots, v_n) = c_1 \land c_2 \land \cdots \land c_r\), over the set \(V = \{v_1, v_2, \ldots, v_n\}\) of variables, construct an instance \((A, >, d)\) of \(\text{MC}_u\text{-MEMBER}\) as follows. The set of alternatives is
\[ A = \{x_i, \bar{x}_i, x'_i, \bar{x}'_i \mid v_i \in V\} \cup \{y_j \mid c_j \text{ is a clause in } \varphi\} \cup \{d\}, \]
where \(d\) is the distinguished alternative whose membership in some minimal upward covering
set for \(A\) is to be decided, and the dominance relation \(>\) is defined by:

- For each \(i, 1 \leq i \leq n\), there is a cycle \(x_i > \bar{x}_i > x'_i > \bar{x}'_i > x_i\);
- if variable \(v_i\) occurs in clause \(c_j\) as a positive literal, then \(x_i > y_j\);
- if variable \(v_i\) occurs in clause \(c_j\) as a negative literal, then \(\bar{x}_i > y_j\); and
- for each \(j, 1 \leq j \leq r\), we have \(y_j > d\).

As an example of this reduction, Figure 2 shows the dominance graph resulting from the
formula
\[ (v_1 \lor \neg v_2 \lor v_3) \land (\neg v_1 \lor \neg v_3), \]
which is satisfiable, for example via the truth assignment that sets each of \(v_1, v_2,\) and \(v_3\) to false. Note that in this case the set \(\{\bar{x}_1, \bar{x}'_1, \bar{x}_2, \bar{x}'_2, \bar{x}_3, \bar{x}'_3\} \cup \{d\}\) is a minimal upward covering set for \(A\) corresponding to the satisfying assignment, so there indeed exists a minimal upward covering
set for \(A\) that contains the designated alternative \(d\). In general, Brandt and Fischer [1] proved
that there exists a satisfying assignment for \( \varphi \) if and only if \( d \) is contained in some minimal upward covering set for \( A \).

As we will use this reduction to prove results for both MC\(_v\)-MEMBER and some of the other problems stated in Section 2, we now analyze the minimal and minimum-size upward covering sets of the dominance graph constructed in the proof sketch of Theorem 4.1. Brandt and Fischer [1] showed that each minimal upward covering set for \( A \) contains exactly two of the four alternatives corresponding to any of the variables, i.e., either \( x_i \) and \( x'_i \), or \( \overline{x_i} \) and \( \overline{x'_i} \), \( 1 \leq i \leq n \). We now assume that if \( \varphi \) is not satisfiable then for each truth assignment to the variables of \( \varphi \), at least two clauses are unsatisfied (which can be ensured, if needed, by adding two dummy variables). It is easy to see that every minimal upward covering set for \( A \) not containing alternative \( d \) must consist of at least \( 2n + 2 \) alternatives where \( 2n \) alternatives are from the variables and at least two, from the unsatisfied clauses. And every minimal upward covering set for \( A \) containing \( d \) consists of exactly \( 2n + 1 \) alternatives, where again \( 2n \) alternatives are from the variables, none from the clauses and alternative \( d \). Thus, \( \varphi \) is satisfiable if and only if every minimum-size upward covering set consists of \( 2n + 1 \) alternatives. These minimum-size upward covering sets always include alternative \( d \).

We now provide another construction that transforms a given boolean formula into a dominance graph with quite different properties.

**Construction 4.2 (for coNP-hardness of upward covering set problems).** Given a boolean formula in conjunctive normal form, \( \varphi(w_1, w_2, \ldots, w_k) = f_1 \land f_2 \land \cdots \land f_l \), over the set \( W = \{w_1, w_2, \ldots, w_k\} \) of variables, we construct a set of alternatives \( A \) and a dominance relation \( \succ \) on \( A \). Without loss of generality, we may assume that if \( \varphi \) is satisfiable then it has at least two satisfying assignments. This can be ensured, if needed, by adding dummy variables.

The set of alternatives is \( A = \{u_i, \overline{u}_i, u'_i, \overline{u'}_i \mid w_i \in W\} \cup \{e_j, e'_j \mid f_j \text{ is a clause in } \varphi\} \cup \{a_1, a_2, a_3\} \), and the dominance relation \( \succ \) is defined by:

- For each \( i, 1 \leq i \leq k \), there is a cycle \( u_i \succ \overline{u}_i \succ u'_i \succ \overline{u'}_i \succ u_i \);
- if variable \( w_i \) occurs in clause \( f_j \) as a positive literal, then \( u_i \succ e_j, u_i \succ e'_j, e_j \succ \overline{u}_i \), and \( e'_j \succ \overline{u}_i \);
- if variable \( w_i \) occurs in clause \( f_j \) as a negative literal, then \( \overline{u}_i \succ e_j, \overline{u}_i \succ e'_j, e_j \succ u_i \), and \( e'_j \succ u_i \);
- if variable \( w_i \) does not occur in clause \( f_j \), then \( e_j \succ u'_i \) and \( e'_j \succ \overline{u'}_i \);
- for each \( j, 1 \leq j \leq \ell \), we have \( a_1 \succ e_j \) and \( a_1 \succ e'_j \); and
- there is a cycle \( a_1 \succ a_2 \succ a_3 \succ a_1 \).

Figure 3 shows some parts of the dominance graph that results from the given boolean formula \( \varphi \). In particular, Figure 3(a) shows that part of this graph that corresponds to some variable \( w_i \) occurring in clause \( f_j \) as a positive literal; Figure 3(b) shows that part of this graph that corresponds to some variable \( w_i \) occurring in clause \( f_j \) as a negative literal; and Figure 3(c) shows that part of this graph that corresponds to some variable \( w_i \) not occurring in clause \( f_j \).
As a more complete example, Figure 4 shows the entire dominance graph that corresponds to the concrete formula \((\neg w_1 \lor w_2) \land (w_1 \lor \neg w_3)\), which can be satisfied by setting, for example, each of \(w_1\), \(w_2\), and \(w_3\) to true. A minimal upward covering set for \(A\) corresponding to this assignment is \(M = \{u_1, u'_1, u_2, u'_2, u_3, u'_3, a_1, a_2, a_3\}\). Note that neither \(e_1\) nor \(e_2\) occurs in \(M\), and none of them occurs in any other minimal upward covering set for \(A\) either. For alternative \(e_1\) in the example shown in Figure 4, this can be seen as follows. If there were a minimal upward covering set \(M'\) for \(A\) containing \(e_1\) (and thus also \(e'_1\), since they both are dominated by the same alternatives) then neither \(u_1\) nor \(u_2\) (which dominate \(e_1\)) must upward cover \(e_1\) in \(M'\), so all alternatives corresponding to the variables \(w_1\) and \(w_2\) (i.e., \(\{u_i, u'_i, u''_i, u'''_i \mid i \in \{1, 2\}\}\)) would also have to be contained in \(M'\). Due to \(e_1 > u'_1\) and \(e'_1 > u''_1\), all alternatives corresponding to \(w_3\) (i.e., \(\{u_3, u'_3, u''_3, u'''_3\}\)) are in \(M'\) as well. Note that, \(e_2\) and \(e'_2\) are no longer upward covered and must also be in \(M'\). The alternatives \(a_1, a_2,\) and \(a_3\) are contained in every minimal upward covering set for \(A\). But then \(M'\) is not minimal because the upward covering set \(M\), which corresponds to the satisfying assignment stated above, is a strict subset of \(M'\). Hence, \(e_1\) cannot be contained in any minimal upward covering set for \(A\).

We now show some properties of the dominance graph created by Construction 4.2 in general. We will need these properties for the proofs in Section 5. The first property, stated in Claim 4.3, has already been seen in the example above.

**Claim 4.3.** Consider the dominance graph \((A, \succ)\) created by Construction 4.2, and fix any \(j\), \(1 \leq j \leq \ell\). For each minimal upward covering set \(M\) for \(A\), if \(M\) contains the alternative \(e_j\)
then all other alternatives are contained in $M$ as well (i.e., $A = M$).

**Proof.** To simplify notation, we prove the claim only for the case of $j = 1$. However, since there is nothing special about $e_1$ in our argument, the same property can be shown by an analogous argument for each $j$, $1 \leq j \leq \ell$.

Let $M$ be any minimal upward covering set for $A$, and suppose that $e_1 \in M$. First note that the dominators of $e_1$ and $e'_1$ are always the same (albeit $e_1$ and $e'_1$ may dominate different alternatives). Thus, for each minimal upward covering set, either both $e_1$ and $e'_1$ are contained in it, or they both are not. Thus, since $e_1 \in M$, we have $e'_1 \in M$ as well.

Since the alternatives $a_1$, $a_2$, and $a_3$ form an undominated three-cycle, they each are contained in every minimal upward covering set for $A$. In particular, $\{a_1, a_2, a_3\} \subseteq M$. Furthermore, no alternative $e_j$ or $e'_j$, $1 \leq j \leq \ell$, can upward cover any other alternative in $M$, because $a_1 \in M$ and $a_1$ dominates $e_j$ and $e'_j$ but none of the alternatives that are dominated by either $e_j$ or $e'_j$. In particular, no alternative in any of the $k$ four-cycles $u_i > u'_i > u_i' > u_i$ can be upward covered by any alternative $e_j$ or $e'_j$, and so they each must be upward covered within their cycle. For each of these cycles, every minimal upward covering set for $A$ must contain at least one of the sets $\{u, u'_i\}$ and $\{u_i, u_i'\}$, since at least one is needed to upward cover the other.
one. 2

Since $e_1 \in M$ and by internal stability, we have that no alternative from $M$ upward covers $e_1$. In addition to $a_1$, the alternatives dominating $e_1$ are $u_i$ (for each $i$ such that $w_i$ occurs as a positive literal in $f_1$) and $\overline{u}_i$ (for each $i$ such that $w_i$ occurs as a negative literal in $f_1$).

First assume that, for some $i$, $w_i$ occurs as a positive literal in $f_1$. Suppose that $\{u_i, u'_i\} \subseteq M$. If $\overline{u}_i' \notin M$ then $e_1 \succ$ would be upward covered by $u_i$, which is impossible. Thus $\overline{u}_i \in M$. But then $u_i \in M$ as well, since $u_i$, the only alternative that could upward cover $\overline{u}_i$, is itself dominated by $\overline{u}_i$. For the latter argument, recall that $\overline{u}_i$ cannot be upward covered by any $e_j$ or $e_j'$. Thus, we have shown that $\{u_i, u'_i\} \subseteq M$ implies $\{\overline{u}_i, \overline{u}_i'\} \subseteq M$. Conversely, suppose that $\{\overline{u}_i, \overline{u}_i'\} \subseteq M$. Then $u'_i$ is no longer upward covered by $\overline{u}_i$ and hence must be in $M$ as well. The same holds for the alternative $u_i$, so $\{\overline{u}_i, \overline{u}_i'\} \subseteq M$ implies $\{u_i, u'_i\} \subseteq M$. Summing up, if $e_1 \in M$ then for each $i$ such that $w_i$ occurs as a positive literal in $f_1$.

By symmetry of the construction, an analogous argument shows that if $e_1 \in M$ then $\{u_i, u'_i, \overline{u}_i, \overline{u}_i'\} \subseteq M$ for each $i$ such that $w_i$ occurs as a negative literal in $f_1$.

Now, consider any $i$ such that $w_i$ does not occur in $f_1$. We have $e_1 \succ u'_i$ and $e'_i \succ \overline{u}_i'$. Again, none of the sets $\{u_i, u'_i\}$ and $\{\overline{u}_i, \overline{u}_i'\}$ alone can be contained in $M$, since otherwise either $u_i$ or $\overline{u}_i$ would remain upward uncovered. Thus, $e_1 \in M$ again implies that $\{u_i, u'_i, \overline{u}_i, \overline{u}_i'\} \subseteq M$.

Now it is easy to see that, since $\bigcup_{1 \leq j \leq k} \{u_j, u'_j, \overline{u}_j, \overline{u}_j'\} \subseteq M$ and since $a_1$ cannot upward cover any of the $e_j$ and $e_j'$, $1 \leq j \leq \ell$, external stability of $M$ enforces that $\bigcup_{1 \leq j \leq \ell} \{e_j, e'_j\} \subseteq M$. Summing up, we have shown that if $e_1$ is contained in any minimal upward covering set $M$ for $A$, then $M = A$.  

\begin{claim}
Consider Construction 4.2. The boolean formula $\varphi$ is satisfiable if and only if there is no minimal upward covering set for $A$ that contains any of the $e_j$, $1 \leq j \leq \ell$.
\end{claim}

\begin{proof}
It is enough to prove the claim for the case $j = 1$, since the other cases can be proven analogously.

From left to right, suppose there is a satisfying assignment $\alpha : W \rightarrow \{0, 1\}$ for $\varphi$. Define the set

$$B_\alpha = \{a_1, a_2, a_3\} \cup \{u_i, u'_i \mid \alpha(w_i) = 1\} \cup \{\overline{u}_i, \overline{u}_i' \mid \alpha(w_i) = 0\}.$$  

Since every upward covering set for $A$ must contain $\{a_1, a_2, a_3\}$ and at least one of the sets $\{u_i, u'_i\}$ and $\{\overline{u}_i, \overline{u}_i'\}$ for each $i$, $1 \leq i \leq k$, $B_\alpha$ is a (minimal) upward covering set for $A$. Let $M$ be an arbitrary minimal upward covering set for $A$. By Claim 4.3, if $e_1$ were contained in $M$, we would have $M = A$. But since $B_\alpha \subset A = M$, this contradicts the minimality of $M$. Thus $e_1 \notin M$.

From right to left, let $M$ be an arbitrary minimal upward covering set for $A$ and suppose $e_1 \notin M$. By Claim 4.3, if any of the $e_j$, $1 < j \leq \ell$, were contained in $M$, it would follow that $e_1 \in M$, a contradiction. Thus, $\{e_j \mid 1 \leq j \leq \ell\} \cap M = \emptyset$. It follows that each $e_j$ must be

\footnote{The argument is analogous to that used in the construction of Brandt and Fischer [1] in their proof of Theorem 4.1. However, in contrast with their construction, which implies that either $\{x_i, x'_i\}$ or $\{\overline{x}_i, \overline{x}_i'\}$, $1 \leq i \leq n$, but not both, must be contained in any minimal upward covering set for $A$ (see Figure 2), our construction also allows for both $\{u_i, u'_i\}$ and $\{\overline{u}_i, \overline{u}_i'\}$ being contained in some minimal upward covering set for $A$. Informally stated, the reason is that, unlike the four-cycles in Figure 2, our four-cycles $u_i > \overline{u}_i > u'_i > \overline{u}_i'$ also have incoming edges.}
upward covered by some alternative in \( M \). It is easy to see that for each \( j, 1 \leq j \leq \ell \), and for each \( i, 1 \leq i \leq k \), \( e_j \) is upward covered in \( M \cup \{ e_j \} \supseteq \{ u_i, u'_i \} \) if \( w_i \) occurs in \( f_j \) as a positive literal, and \( e_j \) is upward covered in \( M \cup \{ e_j \} \supseteq \{ \overline{u}_i, \overline{u}'_i \} \) if \( w_i \) occurs in \( e_j \) as a negative literal. It can never be the case that all four alternatives, \( \{ u_i, u'_i, \overline{u}_i, \overline{u}'_i \} \), are contained in \( M \), because then either \( e_j \) would no longer be upward covered in \( M \) or the resulting set \( M \) was not minimal.

Now, \( M \) induces a satisfying assignment for \( \varphi \) by setting, for each \( i, 1 \leq i \leq k \), \( \alpha(w_i) = 1 \) if \( u_i \in M \), and \( \alpha(w_i) = 0 \) if \( \overline{u}_i \in M \).

Claim 4.5. Consider Construction 4.2. The boolean formula \( \varphi \) is not satisfiable if and only if there is a unique minimal upward covering set for \( A \).

Proof. Recall that we assumed in Construction 4.2 that if \( \varphi \) is satisfiable then it has at least two satisfying assignments.

From left to right, suppose there is no satisfying assignment for \( \varphi \). By Claim 4.4, there must be a minimal upward covering set for \( A \) containing one of the \( e_j \), \( 1 \leq j \leq \ell \), and by Claim 4.3 this minimal upward covering set for \( A \) must contain all alternatives. By reason of minimality, there cannot be another minimal upward covering set for \( A \).

From right to left, suppose there is a unique minimal upward covering set for \( A \). Due to our assumption that if \( \varphi \) is satisfiable then there are at least two satisfying assignments, \( \varphi \) cannot be satisfiable, since if it were, there would be two distinct minimal upward covering sets corresponding to these assignments (as argued in the proof of Claim 4.4).

Wagner provided a sufficient condition for proving \( \Theta^p_2 \)-hardness that was useful in various other contexts (see, e.g., [14, 15, 16, 17, 18]) and is stated here as Lemma 4.6.

Lemma 4.6 (Wagner [14]). Let \( S \) be some \( NP \)-complete problem and let \( T \) be any set. If there exists a polynomial-time computable function \( f \) such that, for all \( m \geq 1 \) and all strings \( x_1, x_2, \ldots, x_{2m} \) satisfying that if \( x_j \in S \) then \( x_{j-1} \in S \), \( 1 < j \leq 2m \), we have
\[
\| \{ i \mid x_i \in S \} \| \text{ is odd} \iff f(x_1, x_2, \ldots, x_{2m}) \in T, \tag{4.1}
\]
then \( T \) is \( \Theta^p_2 \)-hard.

We will apply Lemma 4.6 as well. In contrast with those previous results, however, one subtlety in our construction is due to the fact that we consider not only minimum-size but also (inclusion-)minimal covering sets. To the best of our knowledge, our Construction 4.7 and Construction 4.17, which will be presented later, for the first time apply Wagner’s technique [14] to problems defined in terms of minimality/maximality rather than minimum/maximum size of a solution.\(^3\) In Construction 4.7 below, we define a dominance graph

\(^3\)For example, recall Wagner’s \( \Theta^p_2 \)-completeness result for testing whether the size of a maximum clique in a given graph is an odd number [14]. One key ingredient in his proof is to define an associative operation on graphs, \( \triangleleft \), such
based on Construction 4.2 and the construction presented in the proof sketch of Theorem 4.1 such that Lemma 4.6 can be applied to prove \( MC_\varnothing \)-\( \Theta^C \)-hard (see Theorem 5.2), making use of the properties established in Claims 4.3, 4.4, and 4.5.

**Construction 4.7 (for applying Lemma 4.6 to upward covering set problems).** We apply Wagner’s Lemma with the NP-complete problem \( S = SAT \) and construct a dominance graph. Fix an arbitrary \( m \geq 1 \) and let \( \varphi_1, \varphi_2, \ldots, \varphi_{2m} \) be \( 2m \) boolean formulas in conjunctive normal form such that if \( \varphi_j \) is satisfiable then so is \( \varphi_{j-1} \), for each \( j, 1 < j \leq 2m \). Without loss of generality, we assume that for each \( j, 1 \leq j \leq 2m \), the first variable of \( \varphi_j \) does not occur in all clauses of \( \varphi_j \). Furthermore, we require \( \varphi_j \) to have at least two unsatisfied clauses if \( \varphi_j \) is not satisfiable, and to have at least two satisfying assignments if \( \varphi_j \) is satisfiable. It is easy to see that if \( \varphi_j \) does not have this property, it can be transformed into a formula that does have it, without affecting the satisfiability of the formula.

We now define a polynomial-time computable function \( f \), which maps the given \( 2m \) boolean formulas to a dominance graph \( (A, >) \) with useful properties for upward covering sets. Define \( A = \bigcup_{j=1}^{2m} A_j \) and the dominance relation \( > \) on \( A \) by

\[
\left\{ \begin{array}{l}
\bigcup_{j=1}^{2m} (A_j, >) \\
\bigcup_{i=1}^{m} \left\{ (u_{1,2i}, d_{2i-1}), (\overline{u}'_{1,2i}, d_{2i-1}) \right\} \\
\bigcup_{i=2}^{m} \left\{ (d_{2i-1}, z) \mid z \in A_{2i-2} \right\}
\end{array} \right\},
\]

where we use the following notation:

1. For each \( i, 1 \leq i \leq m \), let \( (A_{2i-1}, >_{2i-1}) \) be the dominance graph that results from the formula \( \varphi_{2i-1} \), according to Brandt and Fischer’s construction [1] given in the proof sketch of Theorem 4.1. We use the same names for the alternatives in \( A_{2i-1} \) as in that proof sketch, except that we attach the subscript \( 2i - 1 \). For example, alternative \( d_1 \) from the proof sketch of Theorem 4.1 now becomes \( d_{2i-1} \), \( x_1 \) becomes \( x_{1,2i-1} \), \( y_1 \) becomes \( y_{1,2i-1} \), and so on.

2. For each \( i, 1 \leq i \leq m \), let \( (A_{2i}, >_{2i}) \) be the dominance graph that results from the formula \( \varphi_{2i} \), according to Construction 4.2. We use the same names for the alternatives in \( A_{2i} \), as in that construction, except that we attach the subscript \( 2i \). For example, alternative \( a_1 \) from Construction 4.2 now becomes \( a_{1,2i} \), \( e_1 \) becomes \( e_{1,2i} \), \( u_1 \) becomes \( u_{1,2i} \), and so on.

3. For each \( i, 1 \leq i \leq m \), connect the dominance graphs \( (A_{2i-1}, >_{2i-1}) \) and \( (A_{2i}, >_{2i}) \) as follows. Let \( u_{1,2i}, \overline{u}_{1,2i}, u'_1, \overline{u}'_{1,2i} \in A_{2i} \) be the four alternatives in the cycle corresponding to the first variable of \( \varphi_{2i} \). Then both \( u'_1 \) and \( \overline{u}'_{1,2i} \) dominate \( d_{2i-1} \). The resulting dominance graph is denoted by \( (B_i, >_i^B) \).

4. Connect the \( m \) dominance graphs \( (B_i, >_i^B), 1 \leq i \leq m \), as follows: For each \( i, 2 \leq i \leq m \), \( d_{2i-1} \) dominates all alternatives in \( A_{2i-2} \).

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that for any two graphs \( G \) and \( H \), the size of a maximum clique in \( G \cong H \) equals the sum of the sizes of a maximum clique in \( G \) and one in \( H \). This operation is quite simple: Just connect every vertex of \( G \) with every vertex of \( H \). In contrast, since minimality for minimal upward covering sets is defined in terms of set inclusion, it is not at all obvious how to define a similarly simple operation on dominance graphs such that the minimal upward covering sets in the given graphs are related to the minimal upward covering sets in the connected graph in a similarly useful way.
The dominance graph \((A, \succ)\) is sketched in Figure 5. Clearly, \((A, \succ)\) is computable in polynomial time.

Before we use this construction to obtain \(\Theta^P_\text{P}-\text{hardness results for some of our upward covering set problems in Section 5, we again show some useful properties of the dominance graph constructed, and we first consider the dominance graph \((B_i, \succ^B_i)\) (see Step 3 in Construction 4.7) separately,\(^4\) for any fixed \(i\) with \(1 \leq i \leq m\). Doing so will simplify our argument for the whole dominance graph \((A, \succ)\). Recall that \((B_i, \succ^B_i)\) results from the formulas \(\varphi_{2i-1}\) and \(\varphi_{2i}\).

**Claim 4.8.** Consider Construction 4.7. Alternative \(d_{2i-1}\) is contained in some minimal upward covering set for \((B_i, \succ^B_i)\) if and only if \(\varphi_{2i-1}\) is satisfiable and \(\varphi_{2i}\) is not satisfiable.

**Proof.** Distinguish the following three cases.

**Case 1:** \(\varphi_{2i-1}\) \(\in\) SAT and \(\varphi_{2i}\) \(\in\) SAT. Since \(\varphi_{2i}\) is satisfiable, it follows from the proof of Claim 4.4 that for each minimal upward covering set \(M\) for \((B_i, \succ^B_i)\), either \([u_{1,2i}, u'_{1,2i}] \subseteq M\) or \([\bar{u}_{1,2i}, \bar{u}'_{1,2i}] \subseteq M\), but not both, and that none of the \(e_{j,2i}\) and \(e'_{j,2i}\) is in \(M\). If \(\bar{u}_{1,2i} \in M\) but \(u'_{1,2i} \notin M\), then \(d_{2i-1} \notin \text{UC}_u(M)\), since \(\bar{u}_{1,2i}\) upward covers \(d_{2i-1}\) within \(M\). If \(u'_{1,2i} \in M\) but \(\bar{u}_{1,2i} \notin M\), then \(d_{2i-1} \notin \text{UC}_u(M)\), since \(u'_{1,2i}\) upward covers \(d_{2i-1}\) within \(M\). Hence, by internal stability, \(d_{2i-1}\) is not contained in \(M\).

**Case 2:** \(\varphi_{2i-1}\) \(\notin\) SAT and \(\varphi_{2i}\) \(\notin\) SAT. Since \(\varphi_{2i-1}\) \(\notin\) SAT, it follows from the proof of Theorem 4.1 that each minimal upward covering set \(M\) for \((B_i, \succ^B_i)\) contains at least one alternative \(y_{j,2i-1}\) (corresponding to some clause of \(\varphi_{2i-1}\)) that upward covers \(d_{2i-1}\). Thus \(d_{2i-1}\) cannot be in \(M\), again by internal stability.

---

\(^4\)Our argument about \((B_i, \succ^B_i)\) can be used to show, in effect, DP-hardness of upward covering set problems, where DP is the class of differences of any two NP sets [35]. Note that DP is the second level of the boolean hierarchy over NP (see Cai et al. [36, 37]), and it holds that NP \(\cup\) coNP \(\subseteq\) DP \(\subseteq\) \(\Theta^P_\text{P}\). Wagner [14] proved appropriate analogs of Lemma 4.6 for each level of the boolean hierarchy. In particular, the analogous criterion for DP-hardness is obtained by using the wording of Lemma 4.6 except with the value of \(m = 1\) being fixed.
**Case 3:** $\varphi_{2i-1} \in \text{SAT}$ and $\varphi_{2i} \notin \text{SAT}$. Since $\varphi_{2i-1} \in \text{SAT}$, it follows from the proof of Theorem 4.1 that there exists a minimal upward covering set $M'$ for $(A_{2i-1}, >_{2i-1})$ that corresponds to a satisfying truth assignment for $\varphi_{2i-1}$. In particular, none of the $y_{j,2i-1}$ is in $M'$. On the other hand, since $\varphi_{2i} \notin \text{SAT}$, it follows from Claim 4.5 that $A_{2i}$ is the only minimal upward covering set for $(A_{2i}, >_{2i})$. Define $M = M' \cup A_{2i}$. It is easy to see that $M$ is a minimal upward covering set for $(B_{i}, >_{i}^B)$, since the only edges between $A_{2i-1}$ and $A_{2i}$ are those from $\bar{u}_{1,2i}$ and $u'_{1,2i}$ to $d_{2i-1}$, and both $\bar{u}_{1,2i}$ and $u'_{1,2i}$ are dominated by elements in $M$ not dominating $d_{2i-1}$.

We now show that $d_{2i-1} \in M$. Note that $\bar{u}_{1,2i}$, $u'_{1,2i}$, and the $y_{j,2i-1}$ are the only alternatives in $B_i$ that dominate $d_{2i-1}$. Since none of the $y_{j,2i-1}$ is in $M$, they do not upward cover $d_{2i-1}$. Also, $u'_{1,2i}$ doesn’t upward cover $d_{2i-1}$, since $\bar{u}_{1,2i} \in M$ and $\bar{u}_{1,2i}$ dominates $u'_{1,2i}$ but not $d_{2i-1}$. On the other hand, by our assumption that the first variable of $\varphi_{2i}$ does not occur in all clauses, there exist alternatives $e_{j,2i}$ and $e'_{j,2i}$ in $M$ that dominate $\bar{u}_{1,2i}$ but not $d_{2i-1}$, so $\bar{u}_{1,2i}$ doesn’t upward cover $d_{2i-1}$ either. Thus $d_{2i-1} \in M$.

Note that, by our assumption on how the formulas are ordered, the fourth case (i.e., $\varphi_{2i-1} \notin \text{SAT}$ and $\varphi_{2i} \in \text{SAT}$) cannot occur. Thus, the proof is complete.

**Claim 4.9.** Consider Construction 4.7. For each $i$, $1 \leq i \leq m$, let $M_i$ be a minimal upward covering set for $(B_i, >_{i}^B)$ according to the cases in the proof of Claim 4.8. Then each of the sets $M_i$ must be contained in every minimal upward covering set for $(A, >)$.

**Proof.** The minimal upward covering set $M_m$ for $(B_m, >_{m}^B)$ must be contained in every minimal upward covering set for $(A, >)$, since no alternative in $A - B_m$ dominates any alternative in $B_m$. On the other hand, for each $i$, $1 \leq i < m$, no alternative in $B_i$ can be upward covered by $d_{2i+1}$ (which is the only element in $A - B_i$ that dominates any of the elements of $B_i$), since $d_{2i+1}$ is dominated within every minimal upward covering set for $B_{i+1}$ (and, in particular, within $M_{i+1}$). Thus, each of the sets $M_i$, $1 \leq i \leq m$, must be contained in every minimal upward covering set for $(A, >)$.

**Claim 4.10.** Consider Construction 4.7. It holds that

$||i \mid \varphi_i \in \text{SAT}||$ is odd $\iff d_1$ is contained in some minimal upward covering set $M$ for $A$.

**Proof.** To show (4.2) from left to right, suppose $||i \mid \varphi_i \in \text{SAT}||$ is odd. Recall that for each $j$, $1 < j \leq 2m$, if $\varphi_j$ is satisfiable then so is $\varphi_{j-1}$. Thus, there exists some $i$, $1 \leq i \leq m$, such that $\varphi_1, \ldots, \varphi_{2i-1} \in \text{SAT}$ and $\varphi_{2i}, \ldots, \varphi_{2m} \notin \text{SAT}$. In Case 3 in the proof of Claim 4.8 we have seen that there is some minimal upward covering set for $(B_i, >_{i}^B)$—call it $M_i$—that corresponds to a satisfying assignment of $\varphi_{2i-1}$ and that contains all alternatives of $A_{2i}$. Note that, $M_i$ contains $d_{2i-1}$. For each $j \neq i$, $1 \leq j \leq m$, let $M_j$ be some minimal upward covering set for $(B_j, >_{j}^B)$ according to Case 1 (if $j < i$) and Case 2 (if $j > i$) in the proof of Claim 4.8.

In Case 1 in the proof of Claim 4.8 we have seen that $d_{2i-3}$ is upward covered either by $\bar{u}_{1,2i-3}$ or by $u'_{1,2i-3}$. This is no longer the case, since $d_{2i-1}$ is in $M_i$, and it dominates all
alternatives in $A_{2i-2}$ but not $d_{2i-3}$. By assumption, $\varphi_{2i-3}$ is satisfiable, so there exists a minimal upward covering set, which contains $d_{2i-3}$ as well. Thus, setting

$$M = \{d_1, d_3, \ldots, d_{2i-3}\} \cup \bigcup_{1 \leq j \leq m} M_j,$$

it follows that $M$ is a minimal upward covering set for $(A, \succ)$ containing $d_1$.

To show (4.2) from right to left, suppose that $||i | \varphi_i \in \text{SAT}||$ is even. For a contradiction, suppose that there exists some minimal upward covering set $M$ for $(A, \succ)$ that contains $d_1$. If $\varphi_1 \notin \text{SAT}$ then we immediately obtain a contradiction by the argument in the proof of Theorem 4.1. On the other hand, if $\varphi_1 \in \text{SAT}$ then our assumption that $||i | \varphi_i \in \text{SAT}||$ is even implies that $\varphi_2 \in \text{SAT}$. It follows from the proof of Claim 4.3 that every minimal upward covering set for $(A, \succ)$ (thus, in particular, $M$) contains either $\{u_{1,2}, u_{1,2}'\}$ or $\{\overline{u}_{1,2}, \overline{u}_{1,2}'\}$, but not both, and that none of the $e_{j,2}$ and $e_j''$ is in $M$. By the argument presented in Case 3 in the proof of Claim 4.8, the only way to prevent $d_1$ from being upward covered by an element of $M$, either $u_{1,2}'$ or $\overline{u}_{1,2}'$, is to include $d_1$ in $M$ as well.\(^5\) By applying the same argument $m - 1$ times, we will eventually reach a contradiction, since $d_{2m-1} \in M$ can no longer be prevented from being upward covered by an element of $M$, either $u_{1,2m}'$ or $\overline{u}_{1,2m}'$. Thus, no minimal upward covering set $M$ for $(A, \succ)$ contains $d_1$, which completes the proof of (4.2).

Furthermore, it holds that $||i | \varphi_i \in \text{SAT}||$ is odd if and only if $d_1$ is contained in all minimum-size upward covering sets for $A$. This is true since the minimal upward covering sets for $A$ that contain $d_1$ are those that correspond to some satisfying assignment for all satisfiable formulas $\varphi_i$, and as we have seen in the analysis of Construction 4.2 and the proof sketch of Theorem 4.1, these are the minimum-size upward covering sets for $A$.

### 4.2. Minimal and Minimum-Size Downward Covering Sets

Turning now to the constructions used to show complexity results about minimal/minimum-size downward covering sets, we again start by giving a proof sketch of a result due to Brandt and Fischer [1], since the following constructions and proofs are based on their construction and proof.

**Theorem 4.11 (Brandt and Fischer [1]).** Deciding whether a designated alternative is contained in some minimal downward covering set for a given dominance graph is NP-hard (i.e., MC-d-Member is NP-hard), even if a downward covering set is guaranteed to exist.

**Proof Sketch.** NP-hardness of MC-d-Member is again shown by a reduction from SAT. Given a boolean formula in conjunctive normal form, $\varphi(v_1, v_2, \ldots, v_n) = c_1 \land c_2 \land \cdots \land c_r$, over the set $V = \{v_1, v_2, \ldots, v_n\}$ of variables, construct a dominance graph $(A, \succ)$ as follows. The set of alternatives is

$$A = \{x_i, \overline{x_i}, x_i', \overline{x_i}', x_{i'}', \overline{x_{i'}}', v_i \in V \} \cup \{y_j, z_j \mid c_j \text{ is a clause in } \varphi\} \cup \{d\},$$

where the membership of alternative $d$ in a minimal downward covering set is to be decided. The dominance relation $\succ$ is defined as follows:

\(^5\)This implies that $d_1$ is not upward covered by either $u_{1,2}'$ or $\overline{u}_{1,2}'$, since $d_1$ dominates them both but not $d_1$. 17
For each $i$, $1 \leq i \leq n$, there is a cycle $x_i \succ x_i' \succ x_i'' \succ x_i$ with two nested three-cycles, $x_i \succ x_i' \succ x_i'' \succ x_i$ and $x_i \succ x_i'' \succ x_i' \succ x_i$;

- If variable $v_i$ occurs in clause $c_j$ as a positive literal, then $y_j \succ x_i$;

- If variable $v_i$ occurs in clause $c_j$ as a negative literal, then $y_j \succ x_i$;

- For each $j$, $1 \leq j \leq r$, we have $d \succ y_j$ and $z_j \succ d$; and

- For each $i$ and $j$ with $1 \leq i, j \leq r$, we have $z_i \succ y_j$.

Brandt and Fischer [1] showed that there is a minimal downward covering set containing $d$ if and only if $\varphi$ is satisfiable. An example of this reduction is shown in Figure 6 for the boolean formula $(v_1 \lor \neg v_2 \lor v_3) \land (\neg v_1 \lor \neg v_3)$. The set $\{x_1, x_1', x_1'', x_2, x_2', x_2'', x_3, x_3', y_1, y_2, z_1, z_2, d\}$ is a minimal downward covering set for the dominance graph shown in Figure 6. This set corresponds to the truth assignment that sets $v_1$ and $v_2$ to true and $v_3$ to false, and it contains the designated alternative $d$.

Regarding their construction sketched above, Brandt and Fischer [1] showed that every minimal downward covering set for $A$ must contain exactly three alternatives for every variable $v_i$ (either $x_i$, $x_i'$, and $x_i''$, or $\overline{x}_i$, $\overline{x}_i'$, and $\overline{x}_i''$), and the undominated alternatives $z_1, \ldots, z_r$. Thus, each minimal downward covering set for $A$ consists of at least $3n + r$ alternatives and induces a truth assignment $\alpha$ for $\varphi$. The number of alternatives contained in any minimal downward covering set for $A$ corresponding to an assignment $\alpha$ is $3n + r + k$, where $k$ is the number of clauses that are satisfied if $\alpha$ is an assignment not satisfying $\varphi$, and where $k = r + 1$ if $\alpha$ is a satisfying assignment for $\varphi$. As a consequence, minimum-size downward covering sets for $A$ correspond to those assignments for $\varphi$ that satisfy the least possible number of clauses of $\varphi$.

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This is different from the case of minimum-size upward covering sets for the dominance graph constructed in the proof sketch of Theorem 4.1. The construction in the proof sketch of Theorem 4.11 cannot be used to obtain...
Next, we provide a different construction to transform a given boolean formula into a dominance graph. This construction will later be merged with the construction from the proof sketch of Theorem 4.11 so as to apply Lemma 4.6 to downward covering set problems.

**Construction 4.12 (for NP- and coNP-hardness of downward covering set problems).**

Given a boolean formula in conjunctive normal form, \( \varphi(w_1, w_2, \ldots, w_k) = f_1 \land f_2 \land \cdots \land f_t \), over the set \( W = \{w_1, w_2, \ldots, w_k\} \) of variables, we construct a dominance graph \((A, \succ)\). The set of alternatives is

\[ A = A_1 \cup A_2 \cup \{a \mid a \in A_1 \cup A_2\} \cup \{b, c, d\} \]

with \( A_1 = \{x_i, x'_i, x''_i, \overline{x}_i, \overline{x}'_i, \overline{x}''_i, z_i, z'_i, z''_i \mid w_i \in W\} \) and \( A_2 = \{y_j \mid f_j \text{ is a clause in } \varphi\} \), and the dominance relation \( \succ \) is defined by:

- For each \( i, 1 \leq i \leq k \), there is, similarly to the construction in the proof of Theorem 4.11, a cycle \( x_i \succ \overline{x}_i \succ x'_i \succ \overline{x}'_i \succ x''_i \succ \overline{x}''_i \succ x_i \) with two nested three-cycles, \( x_i \succ x'_i \succ x''_i \succ x_i \) and \( \overline{x}_i \succ \overline{x}'_i \succ \overline{x}''_i \succ \overline{x}_i \), and additionally we have \( z'_i \succ z_i \succ x_i, z''_i \succ z_i \succ \overline{x}_i, z'_i \succ x_i, z''_i \succ \overline{x}_i \), and \( d \succ z_i \);
- if variable \( w_i \) occurs in clause \( f_j \) as a positive literal, then \( x_i \succ y_j \);
- if variable \( w_i \) occurs in clause \( f_j \) as a negative literal, then \( \overline{x}_i \succ y_j \);
- for each \( a \in A_1 \cup A_2 \), we have \( b \succ \overline{a}, a \succ \overline{a}, \) and \( \overline{a} \succ d \);
- for each \( j, 1 \leq j \leq \ell \), we have \( d \succ y_j \); and
- \( c \succ d \).

An example of this construction is shown in Figure 7 for the boolean formula \((\neg w_1 \lor w_2 \lor w_3) \land (\neg w_2 \lor \neg w_3)\), which can be satisfied by setting for example each of \( w_1, w_2, \) and \( w_3 \) to false. A minimal downward covering set corresponding to this assignment is \( M = \{b, c\} \cup \{\overline{x}_i, \overline{x}'_i, \overline{x}''_i, z'_i, z''_i \mid 1 \leq i \leq 3\} \). Obviously, the undominated alternatives \( b, c, z'_i, \) and \( z''_i \), \( 1 \leq i \leq 3 \), are contained in every minimal downward covering set for the dominance graph constructed. The alternative \( d \), however, is not contained in any minimal downward covering set for \( A \). This can be seen as follows. If \( d \) were contained in some minimal downward covering set \( M' \) for \( A \) then none of the alternatives \( \overline{a} \) with \( a \in A_1 \cup A_2 \) would be downward covered. Hence, all alternatives in \( A_1 \cup A_2 \) would necessarily be in \( M' \), since they all dominate a different alternative in \( M' \). But then \( M' \) is no minimal downward covering set for \( A \), since the minimal downward covering set \( M \) for \( A \) is a strict subset of \( M' \).

We now show some properties of Construction 4.12 in general.

**Claim 4.13.** Minimal downward covering sets are guaranteed to exist for the dominance graph defined in Construction 4.12.
Proof. The set $A$ of all alternatives is a downward covering set for itself. Hence, there always exists a minimal downward covering set for the dominance graph defined in Construction 4.12. 

Claim 4.14. Consider the dominance graph $(A, \succ)$ created by Construction 4.12. For each minimal downward covering set $M$ for $A$, if $M$ contains the alternative $d$ then all other alternatives are contained in $M$ as well (i.e., $A = M$).

Proof. If $d$ is contained in some minimal downward covering set $M$ for $A$, then $\{a, \neg a\} \subseteq M$ for every $a \in A_1 \cup A_2$. To see this, observe that for an arbitrary $a \in A_1 \cup A_2$ there is no $a' \in A$ with $a' \succ a$ and $a' \succ d$ or with $a' \succ a$ and $a' \succ \neg a$. Since the alternatives $c$ and $b$ are undominated, they are also in $M$, so $M = A$.

Claim 4.15. Consider Construction 4.12. The boolean formula $\varphi$ is satisfiable if and only if there is no minimal downward covering set for $A$ that contains $d$. 

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Furthermore, we have that there is no $a$ and $\{d\}$ since there exists a minimal downward covering set for $A$ that does not contain $d$. If there were a minimal downward covering set $M$ for $A$ that contains $d$, Claim 4.14 would imply that $M = A$. However, since $B_a \subset A = M$, this contradicts minimality, so no minimal downward covering set for $A$ can contain $d$.

For the direction from right to left, assume that no minimal downward covering set for $A$ contains $d$. Since by Claim 4.13 minimal downward covering sets are guaranteed to exist for the dominance graph defined in Construction 4.12, there exists a minimal downward covering set $B$ for $A$ that does not contain $d$, so $B \neq A$. It holds that $\{z_i \mid w_i \text{ is a variable in } \varphi \} \cap B = \emptyset$ and $\{y_j \mid f_j \text{ is a clause in } \varphi \} \cap B = \emptyset$, for otherwise a contradiction would follow by observing that there is no $a \in A$ with $a > d$ and $a > z_i$, $1 \leq i \leq k$, or with $a > d$ and $a > y_j$, $1 \leq j \leq \ell$. Furthermore, we have $x_j \notin B$ or $\overline{x}_j \notin B$, for each variable $w_i \in W$. By external stability, for each clause $f_j$ there must exist an alternative $a \in B$ with $a > y_j$. By construction and since $d \notin B$, we must have either $a = x_i$ for some variable $w_i$ that occurs in $f_j$ as a positive literal, or $a = \overline{x}_i$ for some variable $w_i$ that occurs in $f_j$ as a negative literal. Now define $\alpha : W \rightarrow \{0, 1\}$ such that $\alpha(w_i) = 1$ if $x_i \in B$, and $\alpha(w_i) = 0$ otherwise. It is readily appreciated that $\alpha$ is a satisfying assignment for $\varphi$.

\begin{proof}

For the direction from left to right, consider a satisfying assignment $\alpha : W \rightarrow \{0, 1\}$ for $\varphi$, and define the set

$$B_a = \{b, c \} \cup \{x_i, x_i', x_i'' \mid \alpha(w_i) = 1\} \cup \{\overline{x}_i, \overline{x}_i', \overline{x}_i'' \mid \alpha(w_i) = 0\} \cup \{z_i, z_i', z_i'' \mid 1 \leq i \leq k\}.$$ 

It is not hard to verify that $B_a$ is a minimal downward covering set for $A$. Thus, there exists a minimal downward covering set for $A$ that does not contain $d$. If there were a minimal downward covering set $M$ for $A$ that contains $d$, Claim 4.14 would imply that $M = A$. However, since $B_a \subset A = M$, this contradicts minimality, so no minimal downward covering set for $A$ can contain $d$.

\end{proof}

Claim 4.16. Consider Construction 4.12. The boolean formula $\varphi$ is not satisfiable if and only if there is a unique minimal downward covering set for $A$.

\begin{proof}

We again assume that if $\varphi$ is satisfiable, it has at least two satisfying assignments. If $\varphi$ is not satisfiable, there must be a minimal downward covering set for $A$ that contains $d$ by Claim 4.15, and by Claim 4.14 there must be a minimal downward covering set for $A$ containing all alternatives. Hence, there is a unique minimal downward covering set for $A$. Conversely, if there is a unique minimal downward covering set for $A$, $\varphi$ cannot be satisfiable, since otherwise there would be at least two distinct minimal downward covering sets for $A$, corresponding to the distinct truth assignments for $\varphi$, which would yield a contradiction.

\end{proof}

In the dominance graph created by Construction 4.12, the minimal downward covering sets for $A$ coincide with the minimum-size downward covering sets for $A$. If $\varphi$ is not satisfiable, there is only one minimal downward covering set for $A$, so this is also the only minimum-size downward covering set for $A$, and if $\varphi$ is satisfiable, the minimal downward covering sets for $A$ correspond to the satisfying assignments of $\varphi$. As we have seen in the proof of Claim 4.15, these minimal downward covering sets for $A$ always consist of $5k + 2$ alternatives. Thus, they each are also minimum-size downward covering sets for $A$.

Merging the construction from the proof sketch of Theorem 4.11 with Construction 4.12, we again provide a reduction applying Lemma 4.6, this time to downward covering set problems.
Construction 4.17 (for applying Lemma 4.6 to downward covering set problems). We again apply Wagner’s Lemma with the NP-complete problem $S = \text{SAT}$ and construct a dominance graph. Fix an arbitrary $m \geq 1$ and let $\varphi_1, \varphi_2, \ldots, \varphi_{2m}$ be $2m$ boolean formulas in conjunctive normal form such that the satisfiability of $\varphi_j$ implies the satisfiability of $\varphi_{j-1}$, for each $j \in \{2, \ldots, 2m\}$. Without loss of generality, we assume that for each $j$, $1 \leq j \leq 2m$, $\varphi_j$ has at least two satisfying assignments, if $\varphi$ is satisfiable.

We now define a polynomial-time computable function $f$, which maps the given $2m$ boolean formulas to a dominance graph $(A, \succ)$ that has useful properties for our downward covering set problems. The set of alternatives is

$$A = \left( \bigcup_{i=1}^{2m} A_i \right) \cup \left( \bigcup_{i=1}^{m} \{r_i, s_i, t_i\} \right) \cup \{c^*, d^*\},$$

and the dominance relation $\succ$ on $A$ is defined by

$$\left( \bigcup_{i=1}^{2m} \{r_i, d_{2i-1}\}, (r_i, d_{2i}), (s_i, r_i), (s_i, d_{2i-1}), (t_i, r_i), (t_i, d_{2i}) \right) \cup \left( \bigcup_{i=1}^{2m} \{d^*, r_i\} \right) \cup \{(c^*, d^*)\},$$

where we use the following notation:

1. For each $i$, $1 \leq i \leq m$, let $(A_{2i-1}, \succ_{2i-1})$ be the dominance graph that results from the formula $\varphi_{2i-1}$ according to Brandt and Fischer’s construction given in the proof sketch of Theorem 4.11. We again use the same names for the alternatives in $A_{2i-1}$ as in that proof sketch, except that we attach the subscript $2i - 1$.

2. For each $i$, $1 \leq i \leq m$, let $(A_{2i}, \succ_{2i})$ be the dominance graph that results from the formula $\varphi_{2i}$ according to Construction 4.12. We again use the same names for the alternatives in $A_{2i}$ as in that construction, except that we attach the subscript $2i$.

3. For each $i$, $1 \leq i \leq m$, the dominance graphs $(A_{2i-1}, \succ_{2i-1})$ and $(A_{2i}, \succ_{2i})$ are connected by the alternatives $s_i$, $t_i$, and $r_i$ (which play a similar role as the alternatives $z_i$, $z_i''$, and $z_i^*$ for each variable in Construction 4.12). The resulting dominance graph is denoted by $(B_i, \succ_i)$.

4. Connect the $m$ dominance graphs $(B_i, \succ_i)$, $1 \leq i \leq m$ (again similarly as in Construction 4.12). The alternative $c^*$ dominates $d^*$, and $d^*$ dominates the $m$ alternatives $r_i$, $1 \leq i \leq m$.

This construction is illustrated in Figure 8. Clearly, $(A, \succ)$ is computable in polynomial time.

Claim 4.18. Consider Construction 4.17. For each $i$, $1 \leq i \leq 2m$, let $M_i$ be a minimal downward covering set for $(A_i, \succ_i)$. Then each of the sets $M_i$ must be contained in every minimal downward covering set for $(A, \succ)$.

Proof. For each $i$, $1 \leq i \leq 2m$, the only alternative in $A_i$ dominated from outside $A_i$ is $d_i$. Since $d_i$ is also dominated by the undominated alternative $z_{1,i} \in A_i$ for odd $i$, and by the undominated alternative $c_i \in A_i$ for even $i$, it is readily appreciated that internal and external
stability with respect to elements of $A_i$ only depends on the restriction of the dominance graph to $A_i$.

\[ \text{Claim 4.19. Consider Construction 4.17. It holds that} \]
\[ ||\{i \mid \phi_i \in \text{SAT}\}|| \text{ is odd} \]
\[ \iff d^* \text{ is contained in some minimal downward covering set } M \text{ for } A. \quad (4.3) \]

**Proof.** For the direction from left to right in (4.3), assume that $||\{i \mid \phi_i \in \text{SAT}\}||$ is odd. Thus, there is some $j \in \{1, \ldots, m\}$ such that $\phi_1, \phi_2, \ldots, \phi_{2j-1}$ are each satisfiable and $\phi_{2j}, \phi_{2j+1}, \ldots, \phi_{2m}$ are each not. Define
\[ M = \left( \bigcup_{i=1}^{2m} M_i \right) \cup \left( \bigcup_{i=1}^{m} \{s_i, t_i\} \right) \cup \{r_j, c^*, d^*\}, \]
where for each $i$, $1 \leq i \leq 2m$, $M_i$ is some minimal downward covering set of the restriction of the dominance graph to $A_i$, satisfying that $d_i \in M_i$ if and only if

1. $i$ is odd and $\phi_i$ is satisfiable, or
2. $i$ is even and $\phi_i$ is not satisfiable.

Such sets $M_i$ exist by the proof sketch of Theorem 4.11 and by Claim 4.15. In particular, $\phi_{2j-1}$ is satisfiable and $\phi_{2j}$ is not, so $\{d_{2j-1}, d_{2j}\} \subseteq M$. There is no alternative that dominates $d_{2j-1}, d_{2j}$, and $r_j$. Thus, $r_j$ must be in $M$. The other alternatives $r_i$, $1 \leq i \leq m$ and $i \neq j$, are
downward covered by either $s_i$ if $d_{2^i-1} \notin M$, or $t_i$ if $d_{2^i} \notin M$. Finally, $d^*$ cannot be downward covered, because $d^* > r_j$ and no alternative dominates both $d^*$ and $r_j$. Internal and external stability with respect to the elements of $M_i$, as well as minimality of $\bigcup_{i=1}^{2^k} M_i$, follow from the proofs of Theorem 4.11 and Claim 4.15. All other elements of $M$ are undominated and thus contained in every downward covering set. We conclude that $M$ is a minimal downward covering set for $A$ that contains $d^*$.

For the direction from right to left in (4.3), assume that there exists a minimal downward covering set $M$ for $A$ with $d^* \in M$. By internal stability, there must exist some $j$, $1 \leq j \leq k$, such that $r_j \in M$. Thus, $d_{2^j-1}$ and $d_{2^j}$ must be in $M$, too. It then follows from the proof sketch of Theorem 4.11 and Claim 4.15 that $\varphi_{2^j-1}$ is satisfiable and $\varphi_{2^j}$ is not. Hence, $||\{i \mid \varphi_i \in \text{SAT}\}||$ is odd.

By the remark made after Theorem 4.11, Construction 4.17 cannot be used straightforwardly to obtain complexity results for minimum-size downward covering sets.

5. Proof of Theorem 3.1

In this section, we prove Theorem 3.1 by applying the constructions and the properties of the resulting dominance graphs presented in Section 4. We start with the results on minimal and minimum-size upward covering sets.

5.1. Minimal and Minimum-Size Upward Covering Sets

**Theorem 5.1.** It is NP-complete to decide, given a dominance graph $(A, \succ)$ and a positive integer $k$, whether there exists a minimal/minimum-size upward covering set for $A$ of size at most $k$. That is, both $MC_u$-Size and $MSC_u$-Size are NP-complete.

**Proof.** This result can be proven by using the construction of Theorem 4.1. Let $\varphi$ be a given boolean formula in conjunctive normal form, and let $n$ be the number of variables occurring in $\varphi$. Setting the bound $k$ for the size of a minimal/minimum-size upward covering set to $2n+1$ proves that both problems are hard for NP. Indeed, as we have seen in the paragraph after the proof sketch of Theorem 4.1, there is a size $2n+1$ minimal upward covering set (and hence a minimum-size upward covering set) for $A$ if and only if $\varphi$ is satisfiable. Both problems are NP-complete, since they can obviously be decided in nondeterministic polynomial time. $\Box$

**Theorem 5.2.** Deciding whether a designated alternative is contained in some minimal upward covering set for a given dominance graph is hard for $\Theta^p_2$ and in $\Sigma^p_2$. That is, $MC_u$-Member is hard for $\Theta^p_2$ and in $\Sigma^p_2$.

**Proof.** $\Theta^p_2$-hardness follows directly from Claim 4.10. For the upper bound, let $(A, \succ)$ be a dominance graph and $d$ a designated alternative in $A$. First, observe that we can verify in polynomial time whether a subset of $A$ is an upward covering set for $A$, simply by checking whether it satisfies internal and external stability. Now, we can guess an upward covering set $B \subseteq A$ with $d \in B$ in nondeterministic polynomial time and verify its minimality by checking that none of its subsets is an upward covering set for $A$. This places the problem in $\text{NP}^{\text{coNP}}$ and consequently in $\Sigma^p_2$. $\Box$
Theorem 5.3. 1. It is $\Theta_2^p$-complete to decide whether a designated alternative is contained in some minimum-size upward covering set for a given dominance graph. That is, $\text{MSC}_u$-$\text{MEMBER}$ is $\Theta_2^p$-complete.

2. It is $\Theta_2^p$-complete to decide whether a designated alternative is contained in all minimum-size upward covering sets for a given dominance graph. That is, $\text{MSC}_u$-$\text{MEMBER}$-$\text{ALL}$ is $\Theta_2^p$-complete.

Proof. By the remark made after Claim 4.10, both problems are hard for $\Theta_2^p$.

To see that $\text{MSC}_u$-$\text{MEMBER}$ is contained in $\Theta_2^p$, let $(A, >)$ be a dominance graph and $d$ a designated alternative in $A$. Obviously, in nondeterministic polynomial time we can decide, given $(A, >)$, $x \in A$, and some positive integer $\ell \leq ||A||$, whether there exists some upward covering set $B$ for $A$ such that $||B|| \leq \ell$ and $x \in B$. Using this problem as an NP oracle, in $\Theta_2^p$ we can decide, given $(A, >)$ and $d \in A$, whether there exists a minimum-size upward covering set for $A$ containing $d$ as follows. The oracle is asked whether for each pair $(x, \ell)$, where $x \in A$ and $1 \leq \ell \leq ||A||$, there exists an upward covering set for $A$ of size bounded by $\ell$ that contains the alternative $x$. The number of queries is polynomial (more specifically in $O(||A||^2)$), and all queries can be asked in parallel. Having all the answers, determine the size $k$ of a minimum-size upward covering set for $A$, and accept if the oracle answer to $(d, k)$ was yes, otherwise reject.

To show that $\text{MSC}_u$-$\text{MEMBER}$-$\text{ALL}$ is in $\Theta_2^p$, let $(A, >)$ be a dominance graph and $d$ a designated alternative in $A$. We now use as our oracle the set of all $(x, \ell)$, where $x \in A$ is an alternative, and $\ell \leq ||A||$ a positive integer, such that there exists some upward covering set $B$ for $A$ with $||B|| \leq \ell$ and $x \notin B$. Clearly, this problem is also in NP, and the size $k$ of a minimum-size upward covering set for $A$ can again be determined by asking $O(||A||^2)$ queries in parallel (if all oracle answers are no, it holds that $k = ||A||$). Now, the $\Theta_2^p$ machine accepts its input $((A, >), d)$ if the oracle answer for the pair $(d, k)$ is yes, and otherwise it rejects.

Theorem 5.4. 1. (Brandt and Fischer [1]) It is $\text{coNP}$-complete to decide whether a designated alternative is contained in all minimal upward covering sets for a given dominance graph. That is, $\text{MC}_u$-$\text{MEMBER}$-$\text{ALL}$ is $\text{coNP}$-complete.

2. It is $\text{coNP}$-complete to decide whether a given subset of the alternatives is a minimal upward covering set for a given dominance graph. That is, $\text{MC}_u$-$\text{TEST}$ is $\text{coNP}$-complete.

3. It is $\text{coNP}$-hard and in $\Sigma_2^p$ to decide whether there is a unique minimal upward covering set for a given dominance graph. That is, $\text{MC}_u$-$\text{UNIQUE}$ is $\text{coNP}$-hard and in $\Sigma_2^p$.

Proof. It follows from Claim 4.5 that $\varphi$ is not satisfiable if and only if the entire set of alternatives $A$ is a (unique) minimal upward covering set for $A$. Furthermore, if $\varphi$ is satisfiable, there exists more than one minimal upward covering set for $A$ and none of them contains $e_1$ (provided that $\varphi$ has more than one satisfying assignment, which can be ensured, if needed, by adding a dummy variable such that the satisfiability of the formula is not affected). This proves coNP-hardness for all three problems. $\text{MC}_u$-$\text{MEMBER}$-$\text{ALL}$ and $\text{MC}_u$-$\text{TEST}$ are also contained in coNP, as they can be decided in the positive by checking whether there does not exist an upward covering set that satisfies certain properties related to the problem at hand, so they...
both are coNP-complete. MC\textsubscript{u}-Unique can be decided in the positive by checking whether there exists an upward covering set \( M \) such that all sets that are not strict supersets of \( M \) are not upward covering sets for the set of all alternatives. Thus, MC\textsubscript{u}-Unique is in \( \Sigma^p_2 \).

The first statement of Theorem 5.4 was already shown by Brandt and Fischer [1]. However, their proof—which uses essentially the reduction from the proof of Theorem 4.1, except that they start from the coNP-complete problem Validity (which asks whether a given formula is valid, i.e., true under every assignment [21])—does not yield any of the other coNP-hardness results in Theorem 5.4.

**Theorem 5.5.** It is coNP-complete to decide whether a given subset of the alternatives is a minimum-size upward covering set for a given dominance graph. That is, MSC\textsubscript{u}-Test is coNP-complete.

**Proof.** This problem is in coNP, since it can be decided in the positive by checking whether the given subset \( M \) of alternatives is an upward covering set for the set \( A \) of all alternatives (which is easy) and all sets of smaller size than \( M \) are not upward covering sets for \( A \) (which is a coNP predicate), and coNP-hardness follows directly from Claim 4.5.

**Theorem 5.6.** Deciding whether there exists a unique minimum-size upward covering set for a given dominance graph is hard for coNP and in \( \Theta^p_2 \). That is, MSC\textsubscript{u}-Unique is coNP-hard and in \( \Theta^p_2 \).

**Proof.** It is easy to see that coNP-hardness follows directly from the coNP-hardness of MC\textsubscript{u}-Unique (see Theorem 5.4). Membership in \( \Theta^p_2 \) can be proven by using the same oracle as in the proof of the first part of Theorem 5.3. We ask for all pairs \((x, \ell)\), where \( x \in A \) and \( 1 \leq \ell \leq |A| \), whether there is an upward covering set \( B \) for \( A \) such that \( |B| \leq \ell \) and \( x \in B \). Having all the answers, determine the minimum size \( k \) of a minimum-size upward covering set for \( A \). Accept if there are exactly \( k \) distinct alternatives \( x_1, \ldots, x_k \) for which the answer for \((x_i, k), 1 \leq i \leq k\), was yes, otherwise reject.

An important consequence of the proofs of Theorems 5.4 and 5.6 (and of Construction 4.2 that underpins these proofs) regards the hardness of the search problems MC\textsubscript{u}-Find and MSC\textsubscript{u}-Find.

**Theorem 5.7.** Assuming \( P \neq NP \), neither minimal upward covering sets nor minimum-size upward covering sets can be found in polynomial time. That is, neither MC\textsubscript{u}-Find nor MSC\textsubscript{u}-Find are polynomial-time computable unless \( P = NP \).

**Proof.** Consider the problem of deciding whether there exists a nontrivial minimal/minimum-size upward covering set, i.e., one that does not contain all alternatives. By Construction 4.2 that is applied in proving Theorems 5.4 and 5.6, there exists a trivial minimal/minimum-size upward covering set for \( A \) (i.e., one containing all alternatives in \( A \)) if and only if this set is the only minimal/minimum-size upward covering set for \( A \). Thus, the coNP-hardness proof for the problem of deciding whether there is a
unique minimal/minimum-size upward covering set for \( A \) (see the proofs of Theorems 5.4 and 5.6) immediately implies that the problem of deciding whether there is a nontrivial minimal/minimum-size upward covering set for \( A \) is NP-hard. However, since the latter problem can easily be reduced to the search problem (because the search problem, when used as a function oracle, yields the set of all alternatives if and only if this set is the only minimal/minimum-size upward covering set for \( A \)), it follows that the search problem cannot be solved in polynomial time unless \( P = \text{NP} \).

5.2. Minimal and Minimum-Size Downward Covering Sets

**Theorem 5.8.** It is NP-complete to decide, given a dominance graph \((A, \succ)\) and a positive integer \( k \), whether there exists a minimal/minimum-size downward covering set for \( A \) of size at most \( k \). That is, \( \text{MC}_d\text{-Size} \) and \( \text{MSC}_d\text{-Size} \) are both NP-complete.

**Proof.** Membership in NP is obvious, since we can nondeterministically guess a subset \( M \subseteq A \) of the alternatives with \( |M| \leq k \) and can then check in polynomial time whether \( M \) is a downward covering set for \( A \). NP-hardness of \( \text{MC}_d\text{-Size} \) and \( \text{MSC}_d\text{-Size} \) follows from Construction 4.12, the proof of Claim 4.15, and the comments made after Claim 4.16: If \( \varphi \) is a given formula with \( n \) variables, then there exists a minimal/minimum-size downward covering set of size \( 5n + 2 \) if and only if \( \varphi \) is satisfiable.

**Theorem 5.9.** \( \text{MSC}_d\text{-Member} \), \( \text{MSC}_d\text{-Member-All} \), and \( \text{MSC}_d\text{-Unique} \) are coNP-hard and in \( \Theta^p_2 \).

**Proof.** It follows from Claim 4.16 that \( \varphi \) is not satisfiable if and only if the entire set \( A \) of all alternatives is the unique minimum-size downward covering set for itself. Moreover, assuming that \( \varphi \) has at least two satisfying assignments, if \( \varphi \) is satisfiable, there are at least two distinct minimum-size downward covering sets for \( A \). This shows that each of \( \text{MSC}_d\text{-Member} \), \( \text{MSC}_d\text{-Member-All} \), and \( \text{MSC}_d\text{-Unique} \) is coNP-hard. For all three problems, membership in \( \Theta^p_2 \) is shown similarly to the proofs of the corresponding minimum-size upward covering set problems. However, since downward covering sets may fail to exist, the proofs must be slightly adapted. For \( \text{MSC}_d\text{-Member} \) and \( \text{MSC}_d\text{-Unique} \), the machine rejects the input if the size \( k \) of a minimum-size downward covering set cannot be computed (simply because there doesn’t exist any such set). For \( \text{MSC}_d\text{-Member-All} \), if all oracle answers are no, it must be checked whether the set of all alternatives is a downward covering set for itself. If so, the machine accepts the input, otherwise it rejects.

**Theorem 5.10.** It is coNP-complete to decide whether a given subset is a minimum-size downward covering set for a given dominance graph. That is, \( \text{MSC}_d\text{-Test} \) is coNP-complete.

**Proof.** This problem is in coNP, since its complement (i.e., the problem of deciding whether a given subset of the set \( A \) of alternatives is not a minimum-size downward covering set for \( A \)) can be decided in nondeterministic polynomial time. Hardness for coNP follows directly from Claim 4.16.
Theorem 5.11. Deciding whether a designated alternative is contained in some minimal downward covering set for a given dominance graph is hard for $\Theta^p_2$ and in $\Sigma^p_2$. That is, $MC_d$-Member is hard for $\Theta^p_2$ and in $\Sigma^p_2$.

Proof. Membership in $\Sigma^p_2$ can be shown analogously to the proof of Theorem 5.2, and $\Theta^p_2$-hardness follows directly from Claim 4.19.

Theorem 5.12. 1. (Brandt and Fischer [1]) It is coNP-complete to decide whether a designated alternative is contained in all minimal downward covering sets for a given dominance graph. That is, $MC_d$-Member-All is coNP-complete.

2. It is coNP-complete to decide whether a given subset of the alternatives is a minimal downward covering set for a given dominance graph. That is, $MC_d$-Test is coNP-complete.

3. It is coNP-hard and in $\Sigma^p_2$ to decide whether there is a unique minimal downward covering set for a given dominance graph. That is, $MC_d$-Unique is coNP-hard and in $\Sigma^p_2$.

Proof. It follows from Claim 4.16 that $\varphi$ is not satisfiable if and only if the entire set of alternatives $A$ is a unique minimal downward covering set for $A$. Furthermore, if $\varphi$ is satisfiable, there exists more than one minimal downward covering set for $A$ and none of them contains $d$ (provided that $\varphi$ has more than one satisfying assignment, which can be ensured, if needed, by adding a dummy variable such that the satisfiability of the formula is not affected). This proves coNP-hardness for all three problems. $MC_d$-Member-All and $MC_d$-Test are also contained in coNP, because they can be decided in the positive by checking whether there does not exist a downward covering set that satisfies certain properties related to the problem at hand. Thus, they are both coNP-complete. $MC_d$-Unique can be decided in the positive by checking whether there exists a downward covering set $M$ such that all sets that are not strict supersets of $M$ are not downward covering sets for the set of all alternatives. This shows that $MC_d$-Unique is in $\Sigma^p_2$.

The first statement of Theorem 5.12 was already shown by Brandt and Fischer [1]. However, their proof—which uses essentially the reduction from the proof of Theorem 4.11, except that they start from the coNP-complete problem Validity—does not yield any of the other coNP-hardness results in Theorem 5.12.

An important consequence of the proofs of Theorems 5.9 and 5.12 regards the hardness of the search problems $MC_d$-Find and $MSC_d$-Find. (Note that the hardness of $MC_d$-Find also follows from a result by Brandt and Fischer [1, Thm. 9], see the discussion in Section 3.)

Theorem 5.13. Assuming $P \neq NP$, neither minimal downward covering sets nor minimum-size downward covering sets can be found in polynomial time (i.e., neither $MC_d$-Find nor $MSC_d$-Find are polynomial-time computable unless $P = NP$), even when the existence of a downward covering set is guaranteed.
Proof. Consider the problem of deciding whether there exists a nontrivial minimal/minimum-size downward covering set, i.e., one that does not contain all alternatives. By Construction 4.12 that is applied in proving Theorems 5.9 and 5.12, there exists a trivial minimal/minimum-size downward covering set for $A$ (i.e., one containing all alternatives in $A$) if and only if this set is the only minimal/minimum-size downward covering set for $A$. Thus, the coNP-hardness proof for the problem of deciding whether there is a unique minimal/minimum-size downward covering set for $A$ (see the proofs of Theorems 5.9 and 5.12) immediately implies that the problem of deciding whether there is a nontrivial minimal/minimum-size downward covering set for $A$ is NP-hard. However, since the latter problem can easily be reduced to the search problem (because the search problem, when used as a function oracle, yields the set of all alternatives if and only if this set is the only minimal/minimum-size downward covering set for $A$), it follows that the search problem cannot be solved in polynomial time unless $P = NP$. \[\square\]

References


[28] C. Dodgson, A Method of Taking Votes on more than two Issues, pamphlet printed by the Clarendon Press, Oxford, and headed “not yet published” (see the discussions in [38, 39], both of which reprint this paper), 1876.


