Linear Logic and Strong Normalization

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History

- **Girard, TCS '87**: linear logic (LL) and strong normalization (SN).
- A crucial lemma about the exponentials was left **unproven**.
- **Danos, PhD '90**: elaborated proof for second order MELL.
- Various other people worked on **SN for LL**:
  Joinet, van Raamsdonk, Okada, Di Cosmo & Guerrini.
- **Tortora de Falco and Pagani, TCS '10**: SN for second order LL.
- **Complex and long proof**, requiring confluence.
- **Here**: a simple and understandable proof, no need for confluence.

Outline

Kinds of cut

There are **two kinds of cut-elimination cases**.

1) **Principal**, *i.e.* the last rules introduce the cut formulas:

$$\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdots\\ \vdash ?\Gamma, A\end{array}}{\vdash ?\Gamma, !A}\ ! \qquad \dfrac{\begin{array}{c}\theta\\ \vdots\\ \vdash \Delta, A^{\perp}\end{array}}{\vdash \Delta, ?A^{\perp}}\ \mathsf{d}}{\vdash ?\Gamma, \Delta}\ \mathsf{cut} \quad \longrightarrow \quad \dfrac{\begin{array}{c}\pi\\ \vdots\\ \vdash ?\Gamma, A\end{array} \qquad \begin{array}{c}\theta\\ \vdots\\ \vdash \Delta, A^{\perp}\end{array}}{\vdash \Gamma, \Delta}\ \mathsf{cut}$$

2) **Commutative**, one last rule has no relation with the cut formula:

$$\dfrac{\begin{array}{c}\pi\\ \vdots\\ \vdash ?\Gamma, !A\end{array} \qquad \dfrac{\begin{array}{c}\theta\\ \vdots\\ \vdash ?A^{\perp}, ?\Delta, B\end{array}}{\vdash ?A^{\perp}, ?\Delta, !B}\ !}{\vdash ?\Gamma, ?\Delta, !B}\ \mathsf{cut} \quad \longrightarrow \quad \dfrac{\dfrac{\begin{array}{c}\pi\\ \vdots\\ \vdash ?\Gamma, !A\end{array} \qquad \begin{array}{c}\theta\\ \vdots\\ \vdash ?A^{\perp}, ?\Delta, B\end{array}}{\vdash ?\Gamma, ?\Delta, B}\ \mathsf{cut}}{\vdash ?A^{\perp}, ?\Delta, !B}\ !$$

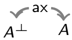
Commutative cases

- Commutative cases are **the burden** of cut-elimination.
- **Problem**: the cut rule commutes with itself.
- **Consequence**: silly diverging reductions.
- **Solution**:
  Switch to proof nets, where commutative cases (mostly) disappear.

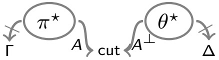
From sequent calculus to proof nets

The **multiplicative fragment**:

$$\frac{}{\vdash A^{\perp}, A}\,\mathsf{ax} \quad \rightsquigarrow$$



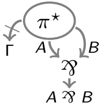
$$\frac{\begin{array}{c}\pi\\ \vdots\\ \vdash \Gamma, A\end{array} \qquad \begin{array}{c}\sigma\\ \vdots\\ \vdash \Delta, A^{\perp}\end{array}}{\vdash \Gamma, \Delta}\,\mathsf{cut} \quad \rightsquigarrow$$



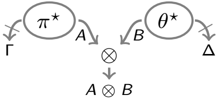
$$\frac{}{\vdash}\,\mathsf{mix}_0 \quad \rightsquigarrow$$

$$\frac{\begin{array}{c}\pi\\ \vdots\\ \vdash \Gamma\end{array} \qquad \begin{array}{c}\sigma\\ \vdots\\ \vdash \Delta\end{array}}{\vdash \Gamma, \Delta}\,\mathsf{mix}_2 \quad \rightsquigarrow$$

$$\frac{\begin{array}{c}\pi\\ \vdots\\ \vdash \Gamma, A, B\end{array}}{\vdash \Gamma, A \parr B}\,\parr \quad \rightsquigarrow$$

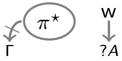


$$\frac{\begin{array}{c}\pi\\ \vdots\\ \vdash \Gamma, A\end{array} \qquad \begin{array}{c}\sigma\\ \vdots\\ \vdash \Delta, B\end{array}}{\vdash \Gamma, \Delta, A \otimes B}\,\otimes \quad \rightsquigarrow$$

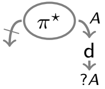


The **exponential fragment**:

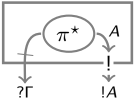
$$\dfrac{\begin{array}{c}\pi\\ \vdots\\ \vdash \Gamma\end{array}}{\vdash \Gamma, ?A}\,\mathsf{w} \quad \rightsquigarrow$$



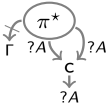
$$\dfrac{\begin{array}{c}\pi\\ \vdots\\ \vdash \Gamma, A\end{array}}{\vdash \Gamma, ?A}\,\mathsf{d} \quad \rightsquigarrow$$



$$\dfrac{\begin{array}{c}\pi\\ \vdots\\ \vdash ?\Gamma, A\end{array}}{\vdash ?\Gamma, !A}\,! \quad \rightsquigarrow$$



$$\dfrac{\begin{array}{c}\pi\\ \vdots\\ \vdash \Gamma, ?A, ?A\end{array}}{\vdash \Gamma, ?A}\,\mathsf{c} \quad \rightsquigarrow$$



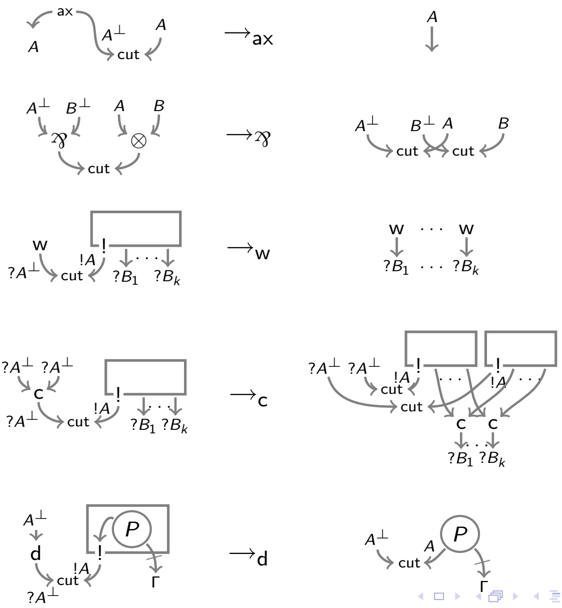
Black-box principle

- Girard introduced boxes according to the **black-box principle**:

  > *"boxes are treated in a perfectly* **modular way***: we can use the box B without knowing its contents, i.e., another box B' with exactly the same doors would do as well"*

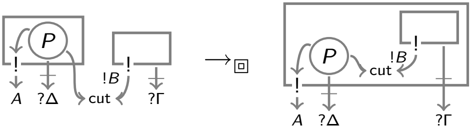
- **Principal cases**: 2 deductive rules cut at level 0 in the same box.
- **Only one commutative case**:
  a rule moving boxes to bring premises of a cut at the same box level

Proof nets cut-elimination: principal cases



Proof nets cut-elimination: the commutative case

Girard's original presentation of proof nets has a commutative case:



This rule is **the source of all technical complications**.

Outline

Exponentials and explicit substitutions

- **Statically**:
  In linear logic $A \Rightarrow B$ decomposes as $!A \multimap B$.
- **Dynamically**:
  β splits in a **multiplicative cut** followed by an **exponential cut**.
- **Intuition**: exponentials = **explicit substitutions**.
- **Ordinary substitution** or **implicit substitution**: $t\{x/s\}$.
- **Explicit substitution**: $t[x/s]$.
- **Then**:

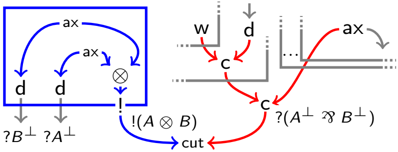
$$(\lambda x.t)s \to_\beta t\{x/s\}$$

becomes

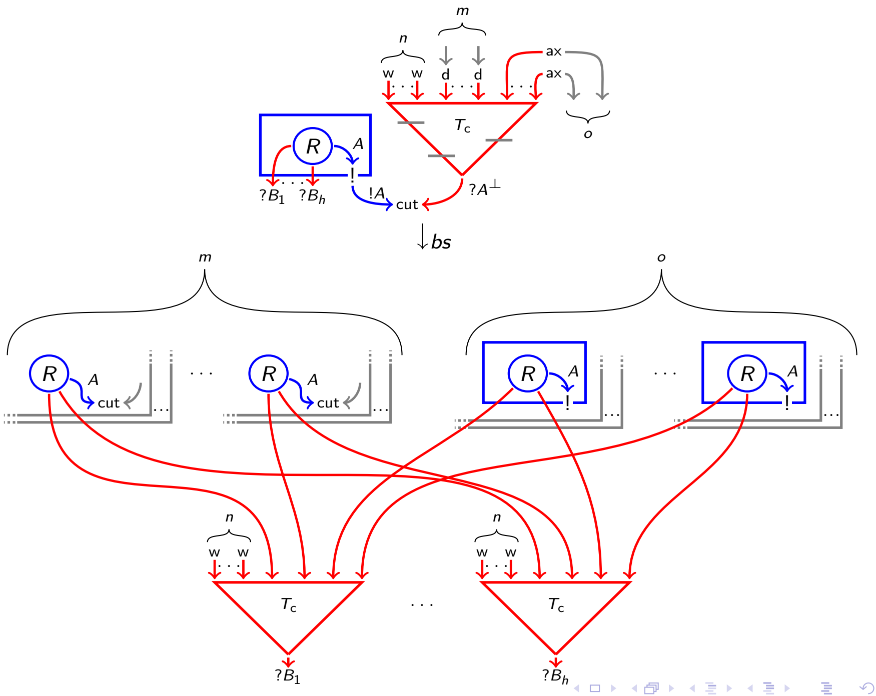
$$(\lambda x.t)s \to_{\mathtt{m}} t[x/s] \to_{\mathtt{e}}^{*} t\{x/s\}$$

What is a variable?

- $[x/s]$ is a **!-box** containing s.
- $t[x/s]$ is a cut between t and the **!-box** around s.
- **What is a variable?** a **maximal tree of ?-rules** (crossing boxes).
- **Example of explicit substitution** $t[x/s]$:



- **Next slide**: definition of substitution in proof nets.


m
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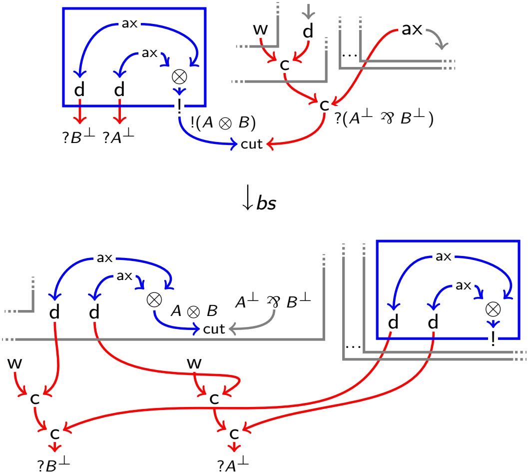
Example of substitution



Outline

The proof technique

- **Proof technique**:
  reducibility candidates in bi-orthogonal form (Girard '87).
- **The proof is axiomatic**:
  it works for **every set of rewriting rules** satisfying the axioms.
- For **Girard's rules** the axioms are **hard to prove**.
- I will later give a **new set of rules** for which the axioms are **easy**.

The axiomatic proof

The proof depends on **3 abstract properties** of the rewriting relation $\rightarrow$:

1. **Substitution and promotion commute**:

$$!(P\{x/Q\}) \rightarrow^* (!P)\{x/Q\}$$

2. **Full composition**:

$$P[x/Q] \rightarrow^+ P\{x/Q\}$$

3. **Kesner's IE property**:

$$\frac{P\{x/Q\} \in SN_{\rightarrow} \qquad Q \in SN_{\rightarrow}}{P[x/Q] \in SN_{\rightarrow}}$$

These properties hold in the **untyped case**.

The IE property

- Key property of λ-calculus:

$$\frac{t\{x/s\}u_1 \dots u_n \in SN_\beta \qquad s \in SN_\beta}{(\lambda x.t)su_1 \dots u_n \in SN_\beta}$$

  called **the fundamental lemma of perpetuality** by van Raamsdonk, Severi, Sorensen, and Xi.

- It is more or less explicitly **used in all proofs of SN**, e.g. van Daalen's for simple types, or Girard's for system F.

- Key point in **inductive definitions of the set of SN λ-terms** (van Raamsdonk & Severi, Loader).

- **Kesner**, LMCS '09:
  Preservation of SN for exp. subst. reduces to the **IE property**:

$$\frac{t\{x/s\}u_1 \dots u_n \in SN_\beta \qquad s \in SN_\beta}{t[x/s]u_1 \dots u_n \in SN_\beta}$$

Key point of the new proof

- The proof is by **induction on the structure** of the net.
- The difficult case is for **promotion**.
- **Inductive Hypothesis**: $!(P[x/Q]) \in SN_{\rightarrow}$ (and $Q \in SN_{\rightarrow}$).
- **Goal**: $(!P)[x/Q] \in SN_{\rightarrow}$.
- **Key point of the proof**:

$$
\begin{array}{rcll}
!(P[x/Q]) & \rightarrow^{+} & !(P\{x/Q\}) \in SN & \text{by full composition and i.h.} \\
 & \rightarrow^{*} & (!P)\{x/Q\} \in SN & \text{by commutation} \\
 & \text{implies} & (!P)[x/Q] \in SN & \text{by the IE property}
\end{array}
$$

- **Novelty**: no analysis of the reducts of $!(P[x/Q])$.

Confluence

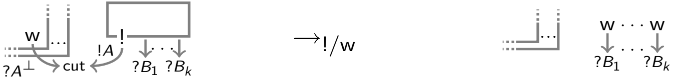
- Main difficulty for the additives: **they are not confluent**.
- **All previous proofs of SN use confluence**.
- That's why T. de Falco and Pagani's proof is **very technical**.
- **Here**: the **first proof** of SN **not requiring confluence**.
- **Consequence**: it smoothly **scales up** to the additives.

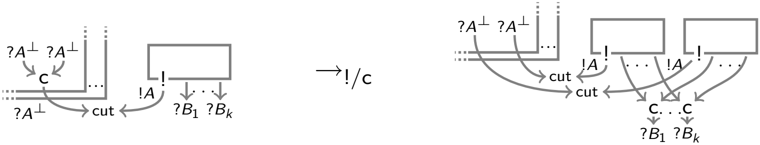
Outline

Black-box principle

- Girard introduced boxes according to the **black-box principle**.
- The black-box principle induces a **commutative case**.
- In such a case **the IE property is hard to prove**.
- **No black-box** in the new approach.
- **Consequences**:
  1. Cuts can be reduced also when they **cross box borders**.
  2. **No commutative case**.
  3. **Easy proof of the IE property**.

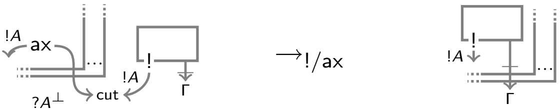
Box-crossing rules 1

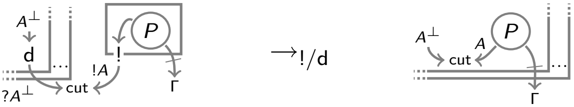




The rules act through possibly many box borders.

Box-crossing rules 2





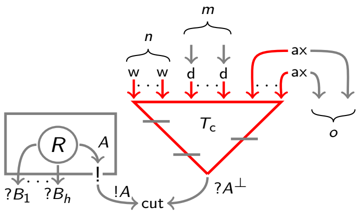
These two cases **absorb the commutative case**.

Proving the IE property

The proof of **the IE property**:

1. box-crossing rules: **two lemmas**, a simple induction on a triple.
2. black-box rules: **many lemmas and pages**, *very* technical.

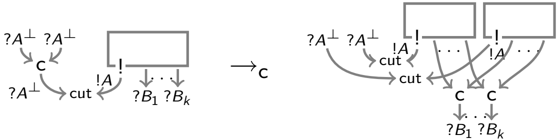
Recall the possible interactions with a graphical variable/?-tree:

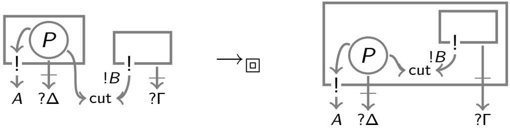


	base cases	inductive cases
box-crossing:	ax, der, weak	contraction
black-box	ax, der, weak	contraction, commutative

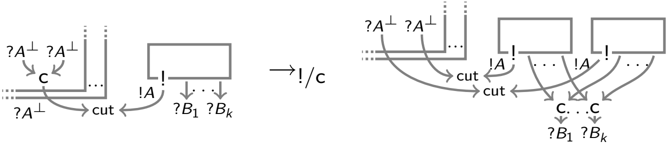
Comparing inductive cases

Black-box rules:





Box-crossing rule:



Intuition: the commutative rule **breaks** the explicit substitution form.

Commutation of promotion and substitution

- The property:

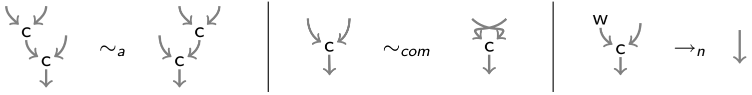
$$!(P\{x/Q\}) \to^* (!P)\{x/Q\}$$

- It follows immediately from the addition of the following rules:

That are **semantically sound** and **needed** to represent λ-terms.

Further extension

- In the paper I also consider the following **optional** rules:



not present in Tortora de Falco and Pagani's proof.

- Usually, their addition requires **delicate and sophisticated reasoning** (Di Cosmo & Guerrini, Tranquilli & Pagani).
- Here it is almost **transparent**.

Ideas

- **Kesner, LMCS '09**:
  IE technique for SN of explicit substitutions.
- **A.-Guerrini, CSL '09**:
  box-free PN for λ-terms with explicit substitutions.
- **A.-Kesner, CSL '10**:
  1. **new approach** to explicit substitutions (structural λ-calculus λ_j).
  2. IE technique **applies extremely easily to λ_j**.
- **Here, RTA '13**:
  back to PN, generalizing Kesner's technique and its application.

Conclusions

Summing up:

1. A neat understanding of **substitution** for proof nets (PN).
2. A simple **axiomatic proof of strong normalization** for LL.
3. A **new presentation of PN** s.t. the axioms are easy to verify.
4. A **new understanding of cut-elimination** and exponential boxes.
5. A **fruitful interaction** between **LL** and **explicit substitutions**.

THANKS!