Generating R.V.s for Simulation via Accept-Reject Method

Big plus: Doesn't require knowing $F_X(x)$ or being able to invert it.
We will focus on the continuous case (discrete is similar).

Goal: Generate X with p.d.f. $f_X(t)$.

Step 1: Find distribution Y with p.d.f. $f_Y(t)$ where

$$f_Y(t) > 0 \Leftrightarrow f_X(t) > 0$$

and where we already know how to generate Y.

Step 2: Let $c \geq 1$: smallest constant s.t.

$$\frac{f_X(t)}{f_Y(t)} \leq c, \qquad \forall t, \text{ s.t. } f_X(t) > 0 \ .$$

Step 3: Generate instance t of Y.

Step 4: With probability $\frac{f_X(t)}{cf_Y(t)}$, accept t, and return $X = t$.
Else, reject t and return to Step 3.

Question: What's the intuition behind step 4?

Theorem: Accept-reject algorithm produces an instance of X, where

$$\mathbf{E}\left[\text{Number needed interations}\right] = c \ .$$

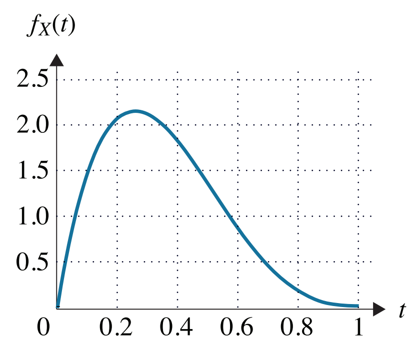
Generating a bounded continuous distribution

Goal: Generate r.v. X s.t.

$$f_X(t) = 20t(1-t)^3, \qquad 0 < t < 1 \ .$$

Question: Why can't we use inverse-transform method?

Question: Looking at $f_X(t)$ below, what's a good suggestion for Y?



Question: What's your algorithm?

Question: How many iterations on avg are needed to get an instance of X?

Generating a Normal distribution

Goal: Generate $N \sim \text{Normal}(0,1)$

$$f_N(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}, \qquad -\infty < t < \infty \; .$$

Question: Is there another r.v. Y that looks like this?

Question: Suppose we instead generate $X = |N|$. How do we convert X to N?

Question: What r.v. Y should we use to generate X?

Generating a Normal distribution, cont.

Question: What is c?

Question: What's our algorithm?

Question: What's the expected number of iterations?

Where we're heading

Goal: Generating arrival process: Poisson Process

Steps:

1. Review of Exponential Distribution
2. Memoryless Distribution
3. Define Poisson Process
4. Properties of Poisson Process

Review of the Exponential Distribution

$X \sim \text{Exp}(\lambda)$.

Question: What is $f_X(t)$?

Question: What is $\overline{F}_X(x) = \mathbf{P}\{X > x\}$?

Question: What is $\mathbf{E}[X]$?

Memoryless Property of the Exponential

Let $X \sim \text{Exp}(\lambda)$. Then:

$$\mathbf{P}\left\{X > s + t \mid X > s\right\} = \mathbf{P}\left\{X > t\right\}, \qquad \forall s, t \geq 0 \ .$$

Question: What does the memoryless property say?

Question: Explain why $X \sim \text{Exp}(\lambda)$ is memoryless.

Question: What is the only discrete-time memoryless distribution?

What is a Poisson process?

Simplest Definition of Poisson Process with rate λ:

Question: Why is this called a “Poisson” process?

Poisson process, cont.

Independent increments & Stationary increments

Question: Why do we care about the Poisson process?

More properties of the Poisson process

Poisson Merging

Poisson Splitting

Uniformity

Practice

Question: A stream of packets arrives according to a Poisson process with avg. rate $\lambda = 50$ packets/sec. What is the expected number of arrivals by time 3 seconds?

Question: A stream of packets arrives according to a Poisson process with avg rate $\lambda = 50$ packets/sec. Suppose each packet is of type "green" with probability 5% and of type "yellow" with probability 95%. Given that 95 green packets arrived during the previous second, what is the expected number of yellow packets that arrives during the previous second?

Question: You are told that by time 30 seconds, 1000 packets have arrived. What is the probability that 200 packets arrived during the first 10 seconds?