15-857/47-774 Homework 5: Ergodicity

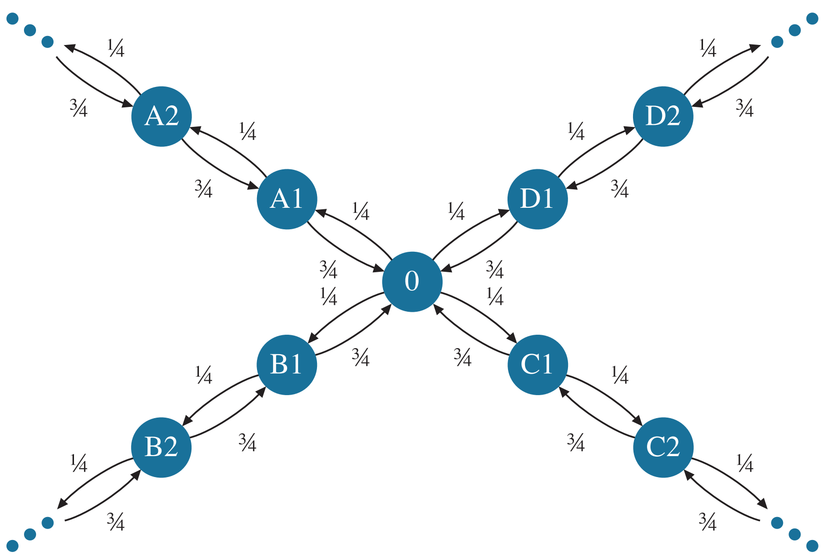
Homework is due at the *start* of Friday's class. You have a full week. We grade your homework right away, so please don't be late. If you're having problems, please go to office hours. Feel free to collaborate with other students, but you should write up your own solutions. It is good form to list the names of people with whom you collaborate.

These problems are from your textbook, *Performance Modeling and Design of Computer Systems*. Starred problems are either not in your textbook, or have some modifications, given below.

<u>Exercises</u>: 9.4 ; 9.31* ; 10.7* ; 10.9* ; 10.10* ; 25.7* ; 25.29*

Exercise 9.31: [Wandering abround the Pittsburgh Airport]

At the Pittsburgh international airport, each of the terminals A, B, C, and D now have an infinite number of gates. A weary traveler in the airport wanders the gates at random, starting from the central hub (0). The traveler's movement is modeled by the Markov chain below.



(a) Is the DTMC time-reversible? Explain *intuitively* why or why not?

(b) Find the stationary distribution $\pi_{Ai}, \pi_{Bi}, \pi_{Ci}, \pi_{Di}$, and π_0.

(c) Find $m_{0,A2}$, the expected time for the traveler to get to their gate $A2$.

Exercise 10.7: [Processor with Failures]

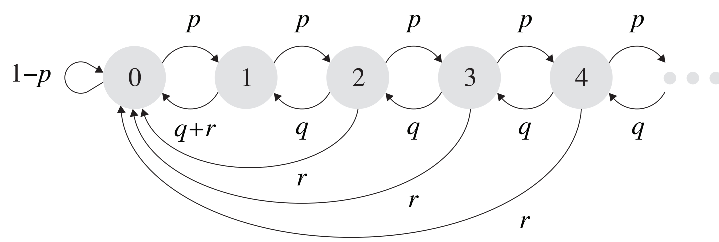


Figure 1: *DTMC for processor with failures.*

The DTMC in Figure 1 is used to model a processor with failures. The chain tracks the number of jobs in the system. At any time step, either the number of jobs increases by 1 (with probability p), or decreases by 1 (with probability q), or a processor failure occurs (with probability r), where $p+q+r=1$. In the case of a processor failure, all jobs in the system are lost. Derive the limiting probability, π_i, of there being i jobs in the system. [This problem is in your book, but I've added steps to help!]

(a) Write the balance equation for state 0. Now express π_1 in terms of π_0.

(b) Write the balance equations for state $i \geq 1$.

(c) Let $\hat{\Pi}(z) = \sum_{i=0}^{\infty} \pi_i z^i$. Derive an expression for $\hat{\Pi}(z)$ in terms of π_0. You should get

$$\hat{\Pi}(z) = \frac{\pi_0 - z\pi_0 - z\frac{r}{q}}{z^2\frac{p}{q} - z\frac{1}{q} + 1}.$$

(d) Rewrite $\hat{\Pi}(z)$ with its denominator factored into $\left(1 - \frac{z}{r_1}\right)\left(1 - \frac{z}{r_2}\right)$, where r_1 and r_2 are roots that you specify, where $r_1 < r_2$.

(e) Determine π_0. You will need three steps:

(i) Explain why $\hat{\Pi}(z)$ is bounded for all $0 \leq z \leq 1$.

(ii) Now show that $0 \leq r_1 < 1$.

iii. We thus can conclude that $\hat{\Pi}(r_1) < \infty$. Thus, since r_1 is a root of the denominator of $\hat{\Pi}(z)$, it must also be a root of the numerator of $\hat{\Pi}(z)$. Use this to get π_0. [Note: Although you now have π_0, wait until the very end of the problem to substitute in this value.]

(f) Apply partial fraction decomposition to $\hat{\Pi}(z)$.

(g) $\hat{\Pi}(z)$ should now be very simple. Rewrite $\hat{\Pi}(z)$ as a geometric series.

(h) Match coefficients to get the π_i's.

(i) Verify that your solution for π_i satisfies the stationary (or balance) equations.

Exercise 10.9: [Hellbound]

[Proposed by Alec Sun] Every lifetime Iggy is reincarnated into either heaven or hell. Since Iggy is a bad boy, reincarnations occur as follows:

- If Iggy is in heaven, then he will always be reincarnated into hell.
- If Iggy is in hell and has been in hell for $j \geq 1$ consecutive lifetimes since last being in heaven, then with probability $0 < p_j < 1$ he is reincarnated into heaven and with probability $1 - p_j$ he is reincarnated into hell.

Figure 2 depicts the infinite-state DTMC showing Iggy's state:

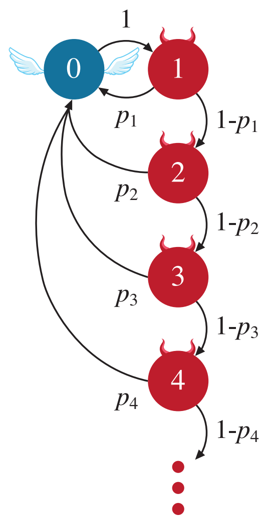


Figure 2: *DTMC for Exercise 10.9.*

(a) Is the DTMC in Figure 2 irreducible, assuming that every value of p_j satisfies $0 < p_j < 1$?

(b) Let $c \in (0, 1)$ be a constant and suppose $p_j = c$ for all $j \geq 1$. Is our DTMC transient, positive recurrent, or null recurrent? Prove your answer.

(c) Suppose $p_j = \frac{1}{j+1}$ for all $j \geq 1$. Is this DTMC transient, positive recurrent, or null recurrent? Prove your answer. [Hint: It may be easier to consider $1 - f_0$.]

(d) Suppose $p_j = 2^{-j}$ for all $j \geq 1$. Is this DTMC transient, positive recurrent, or null recurrent? Prove your answer. [Hint: Compute f_0.]

Exercise 10.10: [How rare are time-reversible DTMCs?]

Edward feels that time-reversible chains are very rare. Erica disagrees. Erica claims that it's easy to create time-reversible chains, via the idea of Exercise 9.8.

(a) Consider the DTMC in Figure 3 whose transitions are unlabeled. Use what you've learned in Exercise 9.8 to label each edge (i, j) of the DTMC with a transition probability p_{ij} such that $0 < p_{ij} < 1$ and such that the DTMC is time-reversible. Then write the *limiting distribution* of your chain.

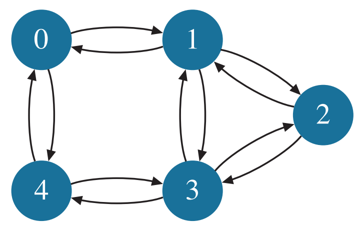


Figure 3: *Markov chain for Exercise 10.10*

(b) How many possible answers are there to question (a)? That is, how many choices of transition probabilities are there that create a time-reversible DTMC? Pick the correct answer and give a one-line explanation:

(i) exactly one

(ii) a finite number

(iii) countably infinite

(iv) uncountably infinite

Exercise 25.7: [A Useful Identity]
In your book. Please add note: X is a non-negative, continuous random variable.

Exercise 25.29: [Total work arriving by time t]
Suppose that the number of jobs which arrive at a datacenter by time t is distributed as Poisson($1000t$). Each job has size (work) which is Exponentially distributed with rate 10. Let W denote the total work that has arrived by time t.

(a) What is $\widetilde{W}(s)$?

(b) Differentiate this to get $\mathbf{E}[W]$. You answer should make sense.