15-857/47-774 Homework 4: DTMCs

Instructions:

Homework is due at the *start* of Friday's class. You have a full week. We grade your homework right away, so please don't be late. If you're having problems, please go to office hours. Feel free to collaborate with other students, but you should write up your own solutions. It is good form to list the names of people with whom you collaborate.

These problems are from your textbook, *Performance Modeling and Design of Computer Systems*. Starred problems are either not in your textbook, or have some modifications, given below. **Note that you'll have to wait until Monday's class for exercises 9.2, 9.8, and 9.17, which deal with time-reversibility.**

<u>Exercises</u>: 8.8*, 9.2, 9.7, 9.8*, 9.17*, 9.23*, 9.30*, 10.1*

Exercise 8.8: [Two time steps per job]
Consider the queue shown in Figure 8.6, however this time imagine that it takes exactly 2 discrete time steps to complete each job. At each time step, a new job arrives with probability $p < 0.5$, independent of the work in the system. At every discrete time step, assuming there is a job in the system, the job at the server gets worked on. Note that if there are 0 jobs in the system, and an arrival occurs, then that arrival will also get worked on during that time step (so it will be half-done at the end of the time step).

(a) Draw the DTMC for this problem.

(b) What is $\mathbf{E}[N]$, the mean number of jobs in the system, as function of p?

Exercise 9.8: [Walk on undirected weighted graph]
This exercise is in your book, but please use the following writeup which is clearer. This problem comes up in many areas. Consider any undirected connected graph with weights: $w_{ij} = w_{ji}$ is the weight on edge (i, j) where $w_{ij} \geq 0$, $\forall i, j$. See for example Figure 1. A particle moves between nodes in a weighted graph as follows: A particle residing at node i will next move to node j with probability P_{ij}, where

$$P_{ij} = \frac{w_{ij}}{\Sigma_j w_{ij}}.$$

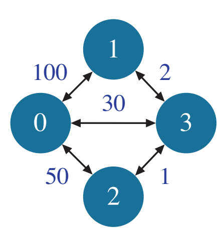

Figure 1: *A weighted graph with $M = 4$ nodes describing a particle's motion.*

Your goal is to determine the long-run proportion of time that the particle is in state i.

(a) Play around with the example in Figure 1. Which node do you think is visited most often?

(b) You'll now need to guess a solution for a *general weighted graph* and show that your solution satisfies the stationary equations. It will help a lot, both in making your guess and in verifying your guess, if you write out the time-reversibility equations rather than the stationary equations.

Exercise 9.17: [Easy Extensions to Theorems in Book]
This problem allows you to argue some very obvious extensions to the theorems in the book that will be helpful in your homework:

(a) In the case of an irreducible, *periodic*, finite-state chain, Section 9.8 tells us that a stationary distribution exists, even though the limiting distribution does not exist. Using the above fact as given, explain why the proof of Theorem 9.6 allows us to make the following additional claim: *For an irreducible,* periodic*, finite-state DTMC,*

$$m_{jj} = \frac{1}{\pi_j}$$

where now π_j is the stationary *probability of being in state j.*

(b) The statement of Theorem 9.34 assumes an aperiodic, irreducible Markov chain. Explain why the proof of Theorem 9.34 allows us to make the following additional claim that obviates ergodicity assumption: *Given a DTMC, if there exist x_1, x_2, x_3, ..., s.t. $\forall i, j$,*

$$\sum_i x_i = 1 \qquad \textit{and} \qquad x_i P_{ij} = x_j P_{ji}$$

then the x_i's are stationary probabilities.

Exercise 9.23: [In an irreducible DTMC, do all states have the same period?]
Given an irreducible DTMC, either prove that all states have the same period, or find a counter-example.

Exercise 9.30: [Irreducible finite-state chains have finite mean time to return]
Prove that for a finite-state, irreducible DTMC, m_{ij} is finite, for every i, j.

Exercise 10.1: [Caching]
Some hints for exercise 10.1 in your book:
1. You will need to think carefully about what information you need in your states to create the appropriate DTMC.
2. When solving your DTMC, you will find that two of the states are only visited a finite number of times, with probability 1, so the long-run fraction of time spent there is 0. You can thus ignore these states and just solve for the stationary probabilities of the remaining states.