15-857/47-774 Homework 2: Convergence, z-Transforms, Little's Law

Instructions:

Homework is due at the *start* of Friday's class. You have a full week. We grade your homework right away, so please don't be late. If you're having problems, please go to office hours. Feel free to collaborate with other students, but you should write up your own solutions. It is good form to list the names of people with whom you collaborate.

These problems are from your textbook, *Performance Modeling and Design of Computer Systems*. Starred problems are either *not* in your textbook or are modified from what's in your textbook. These starred problems are given below.

Exercises:
To do after Wednesday 9/3: 5.1, 5.2*, 5.3*
To do after Friday 9/5: 25.2, 25.20*, 25.21*
To do after Monday 9/8: 6.8*, 6.11*

Replacement for PnC (15-259) students: Note: If you already took PnC (15-259), you may have seen 25.2, 25.20*, and 25.21*. If you don't want to redo these, feel free to write that you've done them and skip them, provided you feel confident about them. As a replacement, however, we'd like you to do **6.7*** on scheduling.

Exercise 5.2: [Two types of convergence]
[Contributed by Weina Wang] Let U be a random variable with the uniform distribution Uniform$(0,1]$. We define a sequence of random variables $\{Y_n : n = 1, 2, 3, \dots\}$ based on U in the following way. Let

$$Y_1 = 1,$$

$$Y_2 = \begin{cases} 1 & \text{if } U \in \left(0, \frac{1}{2}\right] \\ 0 & \text{otherwise} \end{cases} \quad Y_3 = \begin{cases} 1 & \text{if } U \in \left(\frac{1}{2}, 1\right] \\ 0 & \text{otherwise} \end{cases}$$

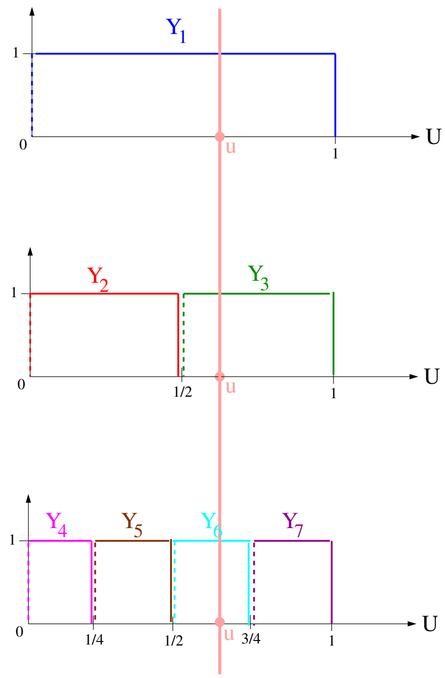
$$Y_4 = \begin{cases} 1 & \text{if } U \in \left(0, \frac{1}{4}\right] \\ 0 & \text{otherwise} \end{cases} \quad Y_5 = \begin{cases} 1 & \text{if } U \in \left(\frac{1}{4}, \frac{1}{2}\right] \\ 0 & \text{otherwise} \end{cases} \quad Y_6 = \begin{cases} 1 & \text{if } U \in \left(\frac{1}{2}, \frac{3}{4}\right] \\ 0 & \text{otherwise} \end{cases} \quad Y_7 = \begin{cases} 1 & \text{if } U \in \left(\frac{3}{4}, 1\right] \\ 0 & \text{otherwise} \end{cases}$$

$$\dots$$

More formally, for each n, let

$$Y_n = \begin{cases} 1 & \text{if } U \in \left(\frac{n - 2^{\lfloor \log_2 n \rfloor}}{2^{\lfloor \log_2 n \rfloor}}, \frac{n - 2^{\lfloor \log_2 n \rfloor} + 1}{2^{\lfloor \log_2 n \rfloor}}\right], \\ 0 & \text{otherwise,} \end{cases}$$

(a) Does $\{Y_n : n = 1, 2, 3, \dots\}$ converge in probability? If so, where does it converge to? Justify your answers.



(b) Does $\{Y_n : n = 1, 2, 3, \ldots\}$ converge almost surely? If so, where does it converge to? Justify your answers.

[Hint: It helps to start by defining formally what a sample path/point, ω, is for this problem.]

Exercise 5.3: [Convergence of minimum of uniform r.v.s]
Let $U_1, U_2, U_3, \ldots$ be i.i.d. random variables with the uniform distribution Uniform$(0, 1)$. We define a sequence of random variables $\{Y_n : n = 1, 2, 3, \ldots\}$ in the following way. For each n, let $Y_n = \min\{U_1, U_2, \ldots, U_n\}$.

(a) Does $\{Y_n : n = 1, 2, 3, \ldots\}$ converge in probability? If so, where does it converge to? Justify your answers.

(b) Does $\{Y_n : n = 1, 2, 3, \ldots\}$ converge almost surely? If so, where does it converge to? Justify your answers.

Exercise 6.7: [More on SRPT]
This exercise is in the book. Just a note about part (b): Assume whatever you want about tie breaks, but do not make that part of your argument.

Exercise 6.8: [Data center utilization]
The Clouds-R-Us company runs a data center with 10,000 servers shown in Figure 1. Jobs arrive to the data center with average rate $\lambda = 2$ jobs/s. Each job requires some number

of servers K, where $K \sim \text{Binomial}(1000, 0.05)$. The job holds onto these K servers for some time S seconds, where $S \sim \text{Exp}(0.02)$, and then releases all its servers at once. Assume that K and S are independent. Jobs are served in FCFS order. If a job gets to the head of the queue, but the number of servers that it needs exceeds the number of idle servers, then the job simply waits (blocking those jobs behind it in the queue) until that number of servers becomes available. You may assume that the system is ergodic. On average, how many jobs are running at a time?

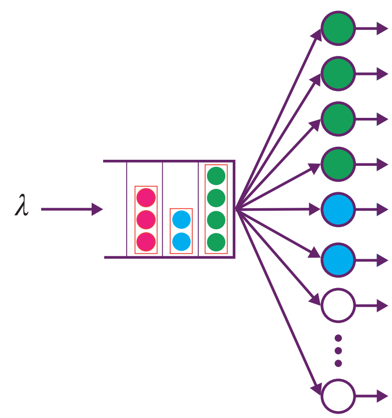


Figure 1: *Data center for Exercise 6.8.*

Exercise 6.11: [Network that looks like a flip flop]

Tianxin's network, shown in Figure 2, looks like a flip flop.

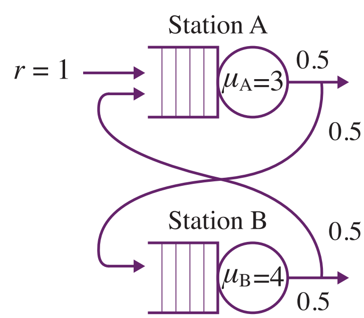


Figure 2: *Tianxin's network for Exercise 6.11.*

Jobs arrive to Tianxin's network at a rate of $r = 1$ jobs per second. The routing probabilities are shown. The service rate at station A is $\mu_A = 3$ jobs per second, and that at station B is $\mu_B = 4$ jobs per second. An individual job might pass through Station A, then B, then A, then B, etc., before it eventually leaves. Tianxin has observed that the expected number of jobs at station A is $\mathbf{E}[N_A] = 2$ and the expected number of jobs at station B is $\mathbf{E}[N_B] = 1$.

(a) Let T denote the response time of a job, i.e., the time from when it arrives until it departs. What is $\mathbf{E}[T]$?

(b) Let λ_A denote the total arrival rate into station A. Let λ_B denote the total arrival rate into station B. What are λ_A and λ_B?

(c) What is the throughput of the system? What is the throughput of station A? Which is higher?

(d) Let T_A denote the time it takes for a job to make a *single* visit to station A (this includes queueing and then serving at station A). Likewise, let T_B denote the time it takes for a job to make a *single* visit to station B. What are $\mathbf{E}[T_A]$ and $\mathbf{E}[T_B]$?

(e) Let T_Q denote the total time that a job spends queueing while in the system. This includes the total time that the job is in queues from when it arrives until it leaves the system. What is $\mathbf{E}[T_Q]$?

Exercise 25.20: [Getting the Distribution from the Transform]
The transform of a random variable captures all moments of the random variable, but does it also capture the distribution? The answer is yes! You are given the z-transform, $\widehat{X}(z)$, of a non-negative, discrete integer-valued random variable, X. Provide an algorithm for extracting the p.m.f. of X from $\widehat{X}(z)$.

Exercise 25.21: [Mouse Trap with Transforms]
A mouse is trapped in a maze. Initially it has to choose one of two directions. If it goes to the right, then it will wander around in the maze for 3 minutes and will then return to its initial position. If it goes to the left, then with probability $\frac{1}{3}$, it will depart the maze after 2 minutes of traveling, and with probability $\frac{2}{3}$ it will return to its initial position after 5 minutes of traveling. Assume that the mouse is at all times equally likely to go to the left or the right. Let T denote the number of minutes that it will be trapped in the maze.

(a) Compute $\mathbf{E}[T]$ (Hint: use conditioning!)

(b) Compute $\mathbf{Var}(T)$ (Hint: again use conditioning but on $\mathbf{E}[T^2]$)

(c) Compute $\widehat{T}(z)$, and then differentiate it to get $\mathbf{E}[T]$, to check your answer.