1 We've seen "Ergodicity Makes Everything Good"

Question: For finite-state chains, what does ergodicity mean?

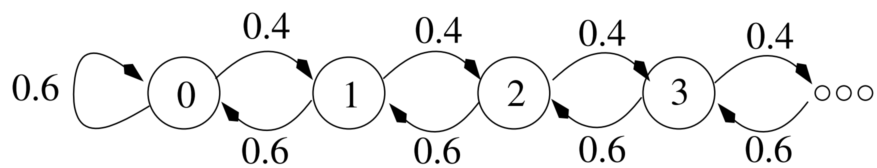
Question: For finite-state chains, what does ergodicity give us?

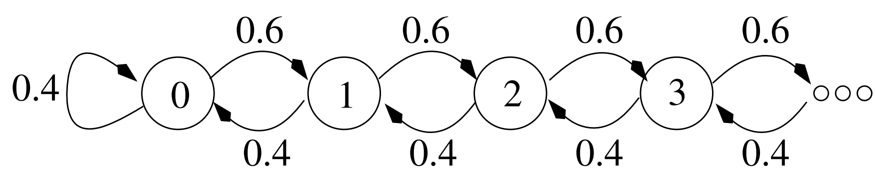
Turns out that the SAME STORY holds for infinite-state chains. Only the definition of "ergodic" needs to be strengthened.

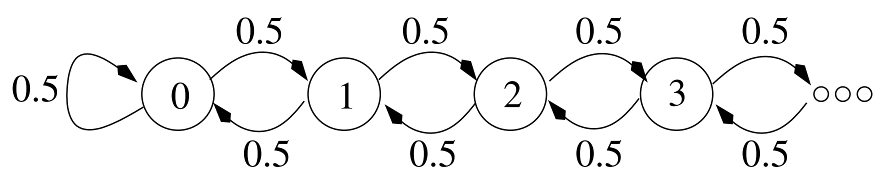
WE WILLL DISCOVER TOGETHER WHAT's NEEDED!

Infinite-state is hard. Proofs are in book. You're only responsible for what we prove in class.

2 Three Infinite-state Chains







QUES: Which of these are aperiodic and irreducible?

QUES: For which DTMCs do you think limiting distribution exists??
Does fish return to shore??

3 Recurrent versus Transient

Defn: f_j is the probability that a chain starting in state j ever returns to state j.

State j is RECURRENT

State j is TRANSIENT

4 Recurrence is a Class Property

Theorem 9.12: If state i is recurrent and i communicates with j, then j is recurrent.

INTUITION:

5 Recurrence is a Class Property, cont.

Theorem 9.12: If state i is recurrent and i communicates with j, then j is recurrent.

FORMAL PROOF:

QUES: Is transience a class property too?

6 Transient chains are bad!

In an irreducible DTMC, either all states are ____________________

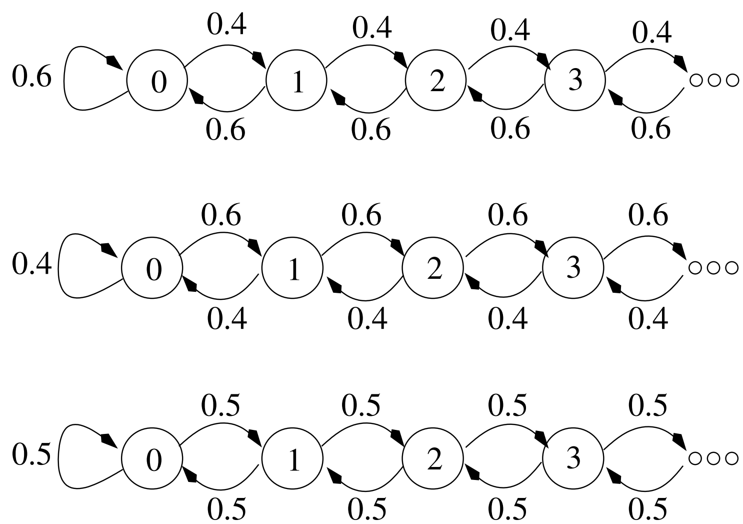
or all are ____________________.

If j: transient then $\pi_j = \lim_{n\to\infty}(\mathbf{P}^n)_{ij} =$ ___________.

If all states are transient, the limiting distribution _____________________.

Theorem 9.17 in book: Also turns out that the stationary distribution does not exist for transient chains.

7 Back to the 3 Chains



QUES: Which are recurrent and which are transient?

QUES: Does the limiting distribution exist for all recurrent, aperiodic, irreducible chains?

8 Commercial Break: Announcements

1. Mor has office hours today: GHC 7207 from 5:30 p.m. - 7 p.m.

2. Keerthana has office hours tomorrow: GHC 7004 from 5:30 p.m. - 7 p.m.

3. Midterm 1: Oct 9th, from 5 p.m. - 7 p.m., in GHC 4307. You will be allowed one 3x5 index card with writing on both sides.

4. How to study:

   (a) Read the book chapters (you should be spending 3 hours after every lecture reading).

   (b) Print blank handouts and try to fill them in yourself. If you want to see my filled-in handout, just drop by my office to take a picture of it or photocopy it.

   (c) Go over all homework solutions, even for problems you got right – extra homework solutions are available outside my office: GHC 7207. Remember that a lot of the theorems come from homework.

   (d) Make a long summary sheet of everything you learned, and then condense it to an index card.

9 Ergodic Theorem of Markov Chains

[**Theorem: 9.25: Ergodic Theorem of Markov Chains**] Given a recurrent, aperiodic, irreducible DTMC, $\pi_j = \lim_{n\to\infty} P_{ij}^n$ exists and

$$\pi_j = \frac{1}{m_{jj}}, \ \forall j.$$

QUES: Wait! Why isn't this enough??

DEFN: Positive-recurrent versus null-recurrent.

Theorem 9.22 of book: Null-recurrence and Positive recurrence are both class properties.

10 Definition of Ergodic

A DTMC is **ergodic** if it has 3 properties:

1. ________________________

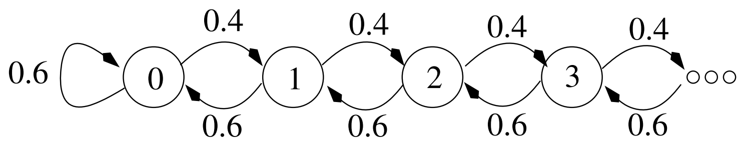
2. ________________________

3. ________________________

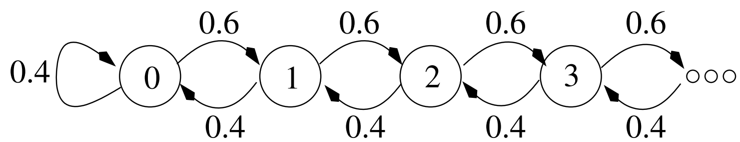
Question: Why didn't we need this 3rd property for finite-state chains?

Question: For ergodic chain, limiting probabilities exist and are positive. But do they sum to 1?

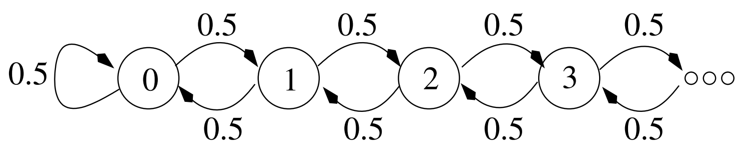
11 Back to the 3 chains: Which are null-recurrent?



(recurrent)



(transient)



(recurrent)

QUES: Which recurrent chain is positive-recurrent? Which is null-recurrent?

12 Summarizing

QUES: What does the Ergodic Thm of Markov Chains say about irreducible + aperiodic + **null-recurrent** chains?

QUES: What does the Ergodic Thm say about irreducible + aperiodic + **positive-recurrent** chains?

13 Summary Theorem

Theorem 9.27 **(Summary Theorem)** An **irreducible & aperiodic** DTMC belongs to *one* of the following two classes:

Either:

(i) All the states are transient, or all are null-recurrent.
In this case $\pi_j = \lim_{n\to\infty} P^n_{ij} =$ __________, $\forall j$,
so the limiting distribution ______________________________________,
and the stationary distribution ______________________________________.

Or:

(ii) All states are positive recurrent.
Then the limiting distribution $\vec{\pi} = (\pi_0, \pi_1, \pi_2, \ldots)$ exists,
and $\pi_j = \frac{1}{m_{jj}} > 0$.
In this case $\vec{\pi}$ is also ____________________, and no other ____________________.

IMPORTANT: Because of the Summary Theorem, you never need to worry about whether your chain is positive recurrent. Simply check for irreducibility and aperiodicity. If these hold, then solve the stationary equations and see if a solution exists. If a stationary distribution exists, then that distribution is also the limiting distribution.