1 Summary so far

Theorems 8.6, 8.8: For any DTMC (finite-state or infinite-state), if the limiting distribution $\vec{\pi}$ exists, then

1. ______________________________________
2. ______________________________________

The ergodicity question: Under what conditions does limiting distribution $\vec{\pi}$ exist?

The plan:

- TODAY: Finish with FINITE-STATE only
- WEDNESDAY: Infinite-state chains

Theorem 9.4 for FINITE-STATE Chains with transition matrix $\mathbf{P}$:

$$\text{Aperiodic \& Irreducible } \implies \mathbf{P}^n \to \mathbf{L}, \text{ as } n \to \infty,$$

where $\mathbf{L}$ is a limiting matrix all of whose rows are the same vector $\vec{\pi}$. The vector $\vec{\pi}$ has all positive components, summing to 1.

For a finite-state DTMC, the word **ergodic** refers to 2 key properties:

___________________ and ___________________

2 Mean Time between Visits to a State in Finite-State Chain

m_{ij} = expected # time steps needed to first get to state j, given we're currently in i.

Lemma: For an irreducible, finite-state Markov chain, $m_{ij} < \infty$, $\forall i, j$. (See HW 4)

Theorem 9.6: Given an irreducible & aperiodic finite-state Markov chain, let π_j denote the limiting probability of being in state j. Then

$$\pi_j = \frac{1}{m_{jj}}.$$

PROOF:

1. Start by writing an expression for m_{ij} and m_{jj}.

2. Let $\mathbf{M}$ be a matrix whose (i, j)th element is m_{ij}.

$$\mathbf{M} = \begin{bmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ m_{20} & m_{21} & m_{22} \end{bmatrix}$$

Let $\mathbf{D}$ be a matrix whose entries are all 0, except for its diagonal entries: $d_{ij} = m_{jj}$.

$$\mathbf{D} = \begin{bmatrix} m_{00} & 0 & 0 \\ 0 & m_{11} & 0 \\ 0 & 0 & m_{22} \end{bmatrix}$$

Let $\mathbf{N}$ be a matrix whose diagonal entries are all 0, but where $N_{ij} = m_{ij}$.

$$\mathbf{N} = \begin{bmatrix} 0 & m_{01} & m_{02} \\ m_{10} & 0 & m_{12} \\ m_{20} & m_{21} & 0 \end{bmatrix}$$

Let $\mathbf{E}$ be a matrix will all entries equal to 1.

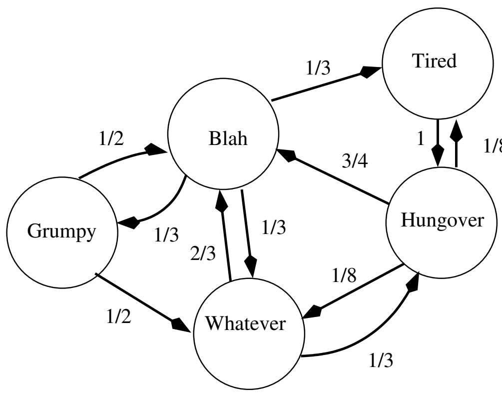
$$\mathbf{E} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Question: Write a matrix equation that expresses your equations from step 1 above:

3. Complete the proof of Theorem 9.6, using your matrix equation from step 2.

3 Long-run Time Average, $\mathbf{p_j}$

Imagine that we have an ergodic, finite-state DTMC:



- Random Walk
- $N_j(t)$

"Time-average" p_j **vs.** **"Ensemble-average"** π_j

BIG QUES: Are p_j and π_j the same?

ANSWER: We'll see YES, w.p.1. Proof requires SLLN & Renewal Theory

4 A bit of renewal theory

Defn: In a **renewal process** the times between events are i.i.d. Let

$$N(t) = \text{Number of events by time } t$$

Renewal Theorem: For a renewal process with mean time between renewals $\mathbf{E}[X] < \infty$,

$$\lim_{t\to\infty} \frac{N(t)}{t} = \qquad\qquad \text{with probability 1}$$

Proof:

5 Ergodicity $\Longrightarrow$ Time-average = Ensemble average

Theorem 9.28 [time-avg = reciprocal of mean-time-to-visit] For a finite-state irreducible DTMC,

$$p_j = \frac{1}{m_{jj}}, \quad \text{w.p. } 1$$

FILL IN PROOF: (NOTE: Aperiodicity NOT needed)

Corollary 9.29 [ergodicity implies time-avg = ensemble avg] For a finite-state aperiodic and irreducible DTMC,

$$p_j = \pi_j, \quad \text{w.p. } 1$$

FILL IN PROOF:

6 Summary: Ergodic Finite-State DTMCs

For a finite-state DTMC, the word **ergodic** refers to 2 key properties:

____________________ and ____________________

For an ergodic finite-state chain:

1. (Thm 9.4) The limiting distribution ______________________________

2. (Thm 9.6) $\pi_j^{limiting} =$ ____

3. (Thm 8.6) The stationary distribution _____________________________

4. (Cor 9.29) The time-average $p_j =$ ______________________________

5. Putting it all together we have that: ______________________________

7 Commercial Break: Announcements

1. Midterm 1: Oct 9th, from 5 p.m. - 7 p.m., in GHC 4307. You will be allowed one 3x5 index card with writing on both sides.

2. How to study:

   (a) Read the book chapters (you should be spending 3 hours after every lecture reading).

   (b) Print blank handouts and try to fill them in yourself. If you want to see my filled-in handout, just drop by my office to take a picture of it or photocopy it.

   (c) Go over all homework solutions, even for problems you got right – extra homework solutions are available outside my office: GHC 7207. Remember that a lot of the theorems come from homework.

   (d) Make a long summary sheet of everything you learned, and then condense it to an index card.

8 Irreducible, but Periodic Finite-State DTMCs

Question: What if your finite-state is irreducible, but has period d?

For an finite-state chain that is irreducible, but periodic:

1. The limiting distribution ______________________________

2. (Sec 9.8.1) The stationary distribution ______________________________

3. (Exercise 9.30, HW 4) m_{jj} is ____________________

4. (Exercise 9.17a, HW 4) $\pi_j^{stationary} =$ ______________________________

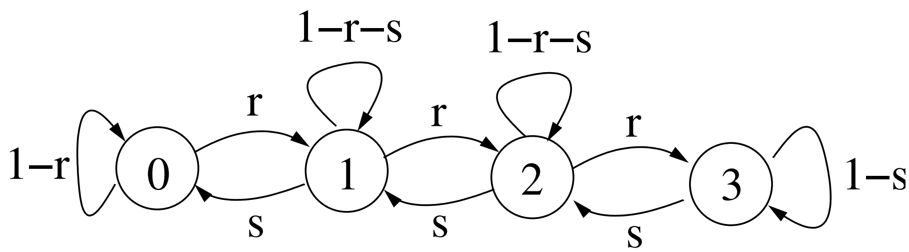
5. (Thm 9.28) The time-average $p_j =$ ______________________________

6. Putting it all together we have that: ______________________________

9 DTMCs that are not irreducible

Question: What happens if your finite-state is not irreducible?

10 An alternative to stationary equations: Time-reversibility equations



Stationary Equations **Time-Reversibility Equations**

$\pi_0 = \pi_0 \cdot (1 - r) + \pi_1 \cdot s$

$\pi_1 = \pi_0 \cdot r + \pi_1 \cdot (1 - r - s) + \pi_2 \cdot s$

$\pi_2 = \pi_1 \cdot r + \pi_2 \cdot (1 - r - s) + \pi_3 \cdot s$

$\pi_3 = \pi_2 \cdot r + \pi_3 \cdot (1 - s)$

$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$

Theorem (Time-Reversibility implies Stationarity)
Given a DTMC with transition matrix $\mathbf{P}$. Suppose we can find $x_0, x_1, x_2, \ldots$ s.t.,

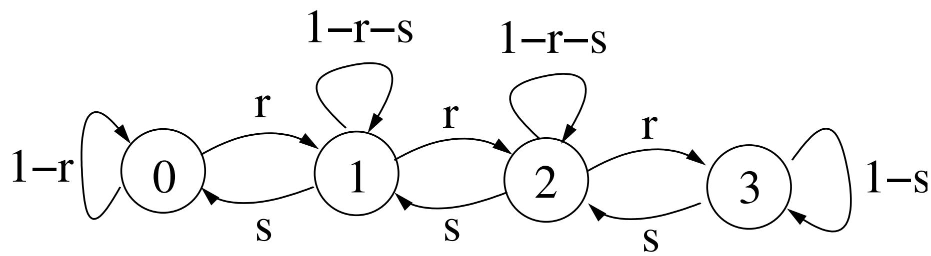
$$x_i P_{ij} = x_j P_{ji} \quad \forall i, j \qquad \text{and} \qquad \sum_i x_i = 1$$

Then the vector $\vec{x} = (x_0, x_1, x_2, \ldots)$ is a stationary distribution, and we say that the DTMC is **time-reversible**.

PROOF: See Exercise 9.17b on HW 4.

Questions on time-reversibility

Question: Why is the "birth-death" process time-reversible?



Questions on time-reversibility

Question: Are all chains time-reversible?

Question: How will I know if chain is time-reversible? When can I use time-reversibility equations?