1 Review

Last class, the emphasis was on finding the limiting distribution.

This class, the emphasis will be on when the limiting distribution exists.

Review: **Stationary distribution** versus **Limiting distribution**

Theorem 8.6: Given a finite-state DTMC with M states:
IF the limiting distrib $\vec{\pi}$ exists, THEN $\vec{\pi}$ is the unique stationary distrib.

Theorem 8.8: Given an infinite-state DTMC:
IF the limiting distrib $\vec{\pi}$ exists, THEN $\vec{\pi}$ is the unique stationary distrib.

2 The three questions of Chpt 9

1. Under what conditions does the limiting distribution exist?

2. What can we say about m_{jj}, the mean time between visits to state j, and how this is related to π_j?

3. How does π_j, the limiting probability of being in state j, compare with p_j, the long-run time-average fraction of time spent in state j?

TODAY: Only **finite-state** chains.

NEXT WEEK: Infinite-state chains.

3 Under what conditions does the Limiting Distribution Exist?

1. Show a 2-state DTMC for which the limiting distribution does not exist. Does the stationary distribution exist? What does it represent?

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4 Definitions

The **period** of state j is ...

State j is **aperiodic** if ...

Markov Chain is **aperiodic** if ...

Question: Why is this the right definition of aperiodic?

5 Definitions, cont.

State j is **accessible** from state i if ...

States i and j **communicate** if ...

A Markov chain is **irreducible** if ...

Question: Is Aperiodicity + Irreducibility sufficient to guarantee limiting distribution exists?

6 THEOREM 9.4

We will now show: For finite state chain:

$$\text{Aperiodic } + \text{ Irreducible } \Longrightarrow \text{Limiting Distribution Exists.}$$

Question: Are aperiodicity and irreducibility *necessary* for the limiting distrib to exist?

Theorem 9.4: Given an aperiodic & irreducible, finite-state DTMC with matrix $\mathbf{P}$.
As $n \to \infty$, $\mathbf{P}^n \to \mathbf{L}$, where $\mathbf{L}$ is a matrix whose rows are all vector $\vec{\pi}$.
The vector $\vec{\pi}$ has all positive components, summing to 1.

EXAMPLE:

$$\mathbf{P} = \begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 1/3 & 1/3 & 1/3 \\ 1/8 & 3/4 & 1/8 \end{pmatrix} \qquad \mathbf{P}^n \to \mathbf{L} = \begin{pmatrix} 0.34 & 0.43 & 0.23 \\ 0.34 & 0.43 & 0.23 \\ 0.34 & 0.43 & 0.23 \end{pmatrix}$$

PROOF: Rephrase Goal in terms of jth column:

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REPHRASE GOAL:

$\vec{e} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, where the 1 is in the jth position. Then STS: ______________________.

INTUITION for why makes sense:

Let M_n denote the Maximum component of $\mathbf{P}^n\vec{e}$.
Let m_n denote the minimum component of $\mathbf{P}^n\vec{e}$.

CLAIM 1: Let s be smallest element in $\mathbf{P}$. Then

$$M_n - m_n \leq (1-2s)\,(M_{n-1} - m_{n-1})\,.$$

PROOF:

1. Upper bound on M_n:

2. Lower bound on m_n:

Question: The proof is complete except for small hole. What's the hole? How to fix it?

<u>The hole</u>:

<u>The fix</u>:

<u>Why the fix completes the proof</u>:

Commercial Break: Announcements

1. NO CLASS this FRIDAY!

2. DROP OFF HW3 at Mor's Office (GHC 7207) at 2 pm on Friday.

3. Mor has office hours today: GHC 7207, from 5:30 p.m. - 7 p.m.

4. Schedule a time to talk to me about a queueing application. If you can't think of any queueing problem, then look for a Markov Chain application.

CLAIM 2:
$\mathbf{P}$: aperiodic, irreducible $\Longrightarrow \exists n_0$, s.t., $\forall n \geq n_0$, $\mathbf{P}^n$ has only positive elts.

PROOF:

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$\mathbf{P}$: aperiodic, irreducible $\Longrightarrow \exists n_0$, s.t., $\forall n \geq n_0$, $\mathbf{P}^n$ has only positive elts.

PROOF cont.:

7 Mean Time between Visits in Finite-State Chain

m_{ij} = expected # time steps needed to first get to state j, given we're currently in i.

Lemma: For an irreducible, finite-state Markov chain, $m_{ij} < \infty$, $\forall i, j$. (See HW 4)

Theorem 9.6: For an irreducible, aperiodic finite-state Markov chain. Let $\pi_j = \lim_{n\to\infty}(\mathbf{P})^n_{ij}$. Then

$$\pi_j = \frac{1}{m_{jj}}.$$

PROOF:

1. Start by writing an expression for m_{ij} and m_{jj}.

2. Let $\mathbf{M}$ be a matrix whose (i, j)th element is m_{ij}.

$$\mathbf{M} = \begin{bmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ m_{20} & m_{21} & m_{22} \end{bmatrix}$$

Let $\mathbf{D}$ be a matrix whose entries are all 0, except for its diagonal entries: $d_{ij} = m_{jj}$.

$$\mathbf{D} = \begin{bmatrix} m_{00} & 0 & 0 \\ 0 & m_{11} & 0 \\ 0 & 0 & m_{22} \end{bmatrix}$$

Let $\mathbf{N}$ be a matrix whose diagonal entries are all 0, but where $N_{ij} = m_{ij}$.

$$\mathbf{N} = \begin{bmatrix} 0 & m_{01} & m_{02} \\ m_{10} & 0 & m_{12} \\ m_{20} & m_{21} & 0 \end{bmatrix}$$

Let $\mathbf{E}$ be a matrix will all entries equal to 1.

$$\mathbf{E} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Q: Write a matrix equation for your equations from step 1:

3. Complete proof of Thm 9.6, using the matrix equation from step 2.