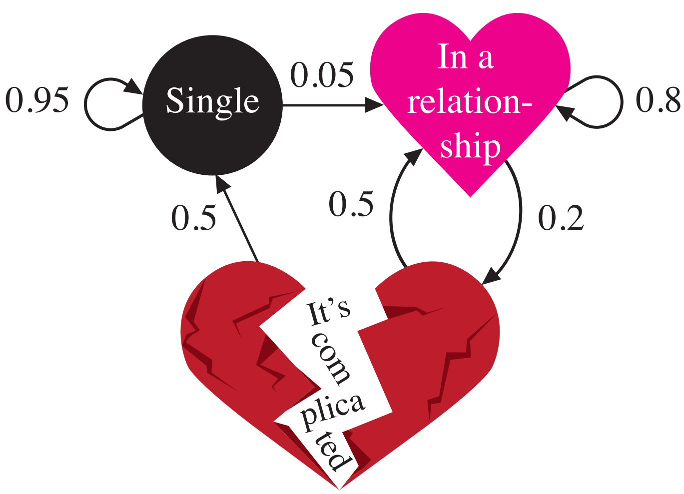
Operation Laws were great for bounding performance in **closed** systems.

But we don't have anything like that for **open** systems. We therefore turn to a new tool: Markov chains.

## 1 Let's talk about Love




**Question:** What is the transition probability matrix, **P**?

The question of life: ____________________

## 2 Discrete-Time Markov Chains

<u>Definition</u>: A **DTMC** (Discrete-Time Markov Chain) is a stochastic process $\{X_n, n = 0, 1, 2, \ldots\}$, s.t.

- $X_n$ denotes ______________________________
- $\forall n \geq 0$, $\forall$ integers $i, j$, and $\forall$ integers $i_0, ..., i_{n-1}$,

$$\mathbf{P}\left\{X_{n+1} = j \mid X_n = i,\ X_{n-1} = i_{n-1},\ ..., X_0 = i_0\right\}$$

$$= \underline{\hspace{8cm}}$$

$$= \underline{\hspace{8cm}}$$

where $P_{ij}$ is independent of the time step and of past history.

**Outline:**

1. Understanding limiting distributions for DTMCs (**Chpt 8**)
2. Ergodicity – existence of limiting distributions (**Chpt 9**)
3. Real-world applications of DTMCs (**Chpt 10**)
4. Poisson Process (**Chpt 11**)
5. CTMCs – Continuous-Time Markov Chains (**Chpt 12**)

# 3 Powers of P

$$\mathbf{P} = \begin{bmatrix} \quad & \quad & \quad \\ \\ \\ \end{bmatrix} \qquad \mathbf{P}^5 = \begin{bmatrix} \quad & \quad & \quad \\ \\ \\ \end{bmatrix} \qquad \mathbf{P}^{50} = \begin{bmatrix} \quad & \quad & \quad \\ \\ \\ \end{bmatrix}$$

<u>Two Observations</u>:

(i) All rows are the same for high powers of $\mathbf{P}$. Why?

(ii) Entries within a row always sum to 1. Why?

1. What does $\left(\mathbf{P}^2\right)_{ij}$ represent?

2. What does $(\mathbf{P}^n)_{ij}$ represent?

## 4 Definition of Limiting Distribution ($M$-state chain)

- Definition of **limiting probability** $\pi_j$:

- Definition of **limiting distribution** $\vec{\pi}$:

- **Question:** How do we get limiting distribution?

- **Question:** What is $\pi_{\text{single}}$?

## 5 Definition of Stationary Distribution ($M$-state chain)

Defn: A probability distribution $\vec{\pi} = (\pi_0, \pi_1, \ldots, \pi_{M-1})$
is said to be **stationary** for a DTMC with transition matrix $\mathbf{P}$ if:

The **stationary equations** are:

**Question:** Why is $\vec{\pi}$ called stationary?

# 6 Stationary distribution = Limiting distribution

**Theorem**: (Stationary=Limiting) Given a finite-state DTMC with $M$ states, let

$$\pi_j = \lim_{n \to \infty} P_{ij}^n$$

Let the limiting distribution be:

$$\vec{\pi} = (\pi_0, \pi_1, \ldots, \pi_{M-1}), \quad \text{where } \sum_{i=0}^{M-1} \pi_i = 1 \ .$$

Assuming that the limiting distribution exists, then $\vec{\pi}$ is also a stationary distribution and *no other* stationary distribution exists.

---

**Question:** What's the impact of the above theorem?

**Question:** What's the intuition behind the theorem?

**PROOF STRUCTURE**: We will show 2 things:

1. Show $\vec{\pi} = (\pi_0, \pi_1, \ldots, \pi_{M-1})$ is a stationary distribution. Hence at least one stationary distribution exists.

2. Show that any stationary distribution, $\vec{\pi}'$, must be equal to the limiting distribution, $\vec{\pi}$.

# 7 Stationary distribution = Limiting distribution

**Theorem**: Given a finite-state DTMC with $M$ states, let

$$\pi_j = \lim_{n \to \infty} P_{ij}^n$$

Let the limiting distribution be:

$$\vec{\pi} = (\pi_0, \pi_1, \ldots, \pi_{M-1}), \quad \text{where } \sum_{i=0}^{M-1} \pi_i = 1 \ .$$

Assuming that the limiting distribution exists, then $\vec{\pi}$ is also a stationary distribution and *no other* stationary distribution exists.

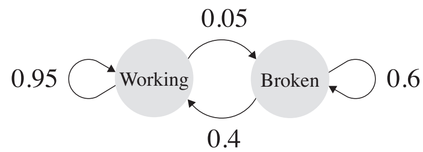
---

**Proof**:

1. Show $\vec{\pi} = (\pi_0, \pi_1, \ldots, \pi_{M-1})$ is a stationary distribution.

2. Show any stationary distribution, $\vec{\pi}'$, must equal the limiting distrib.

# 8 Solving Stat. Eqns to get Limiting Distribution

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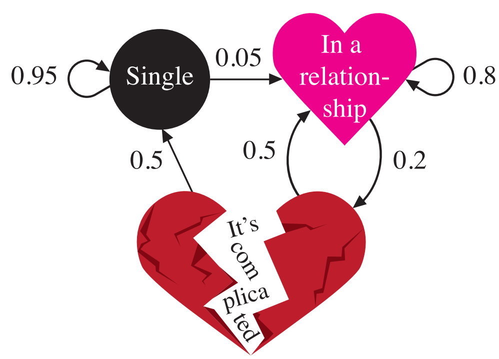

# 9 Commercial Break: Announcements

1. HW 2 returned today. Each problem was given 0, 1, 2, 3 points. Total: 24. Scores were high.

2. Please read **solutions** for all problems, not just the ones that were marked wrong!

3. See TAs if there's a problem, but try not to obsess over grades. Focus on learning!

4. HW 3 is due on Friday.

5. **Zhouzi has office hours TODAY**: **3:30 p.m. - 5 p.m.** in **GHC 6003**. You can ask Zhouzi more questions about his cool talk last Friday!

6. Please keep up with the reading for this class. Spend 3 hours after every class reading the relevant chapter. If you miss a class, you can see me to copy my notes.

## 10 Infinite-state DTMCs

**Question:** How should we define the limiting distribution?

**Question:** How should we define the stationary distribution?

---

Theorem: (Stationary = Limiting) Given an infinite-state DTMC, let

$$\pi_j = \lim_{n \to \infty} (\mathbf{P}^n)_{ij}$$

be the limiting probability of being in state $j$ (independent of the starting state $i$) and let

$$\vec{\pi} = (\pi_0, \pi_1, \pi_2, \ldots), \quad \text{where } \sum_{i=0}^{\infty} \pi_i = 1$$

be the limiting distribution. Assuming that the limiting distribution exists, then $\vec{\pi}$ is also a stationary distribution and *no other* stationary distribution exists.

**Proof:** See book for full proof, but here are a few steps ...

# 11 Modeling problem with Infinite State DTMC

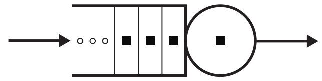

Figure 1: *Illustration of a router with unbounded buffer.*

At every time step, with probability $p = \frac{1}{4}$ one packet arrives, and independently, with probability $q = \frac{1}{3}$ one packet departs. Note that during a time step, we might have both an arrival and a transmission, or neither. A packet can "arrive" and "depart" within the same timestep.

**Goal: Average number of packets in the system**

# Writing the Stationary Equations

0:

1:

2:

3:

## Solving Stationary eqns to get Limiting Distrib

## Deriving the mean number $\mathbf{E}[N]$

# 12 Challenge Problem

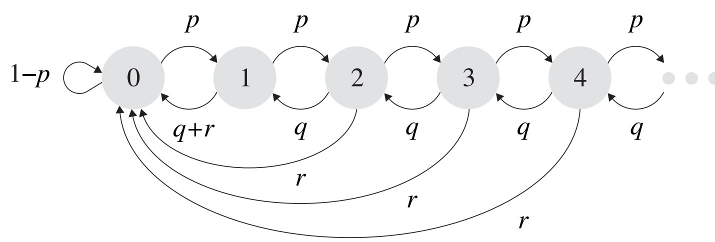


Figure 2: *DTMC for processor with failures.*