1 Today: More Vocabulary

PART I: **Two Types of Convergence**

(a) Convergence in probability

(b) Convergence almost surely (a.k.a., convergence with probability 1)

PART II: **Two Types of Averages**

(a) Time Average

(b) Ensemble Average

Caveats:

- I'm only covering what we need for this course, not doing comprehensive coverage.
- Emphasis is on intuition.
- HW 2 will have examples.

2 Convergence of a Sequence of Constants: $a_n \to b$

<u>Definition:</u>

$\{a_n : n = 1, 2, 3, \ldots\}$ converges to b as $n \to \infty$
if
$\forall$ ________________, $\exists$ ________________, s.t. $\forall$ ________________,

$$|a_n - b| < \epsilon.$$

PICTURE:

3 Convergence of a Sequence of R.V.s: $Y_n \to \mu$

$\{Y_n : n = 1, 2, 3, \ldots\}$: sequence of r.v.s

Want to say that the sequence converges to a <u>constant</u>.

KEY INSIGHT: A r.v. becomes a constant, once we specify the outcome, ω, of the experiment.

EXAMPLE:

- X = r.v. = sum of 2 rolls of a die
- ω = Outcome of experiment = $(2, 3)$
- $X(\omega)$ = __________ = constant!

EXAMPLE:

- $\{Y_n : n = 1, 2, 3, \ldots\}$ = sequence of r.v.s
- Y_n = Avg of first n tosses of fair coin
- ω = Outcome of experiment = 0 0 1 0 1 1 1 0 1 0 . . .
- $\{Y_n(\omega)\}$ = ____

4 Convergence Almost Surely: $Y_n \xrightarrow{a.s.} \mu$

$\{Y_n : n = 1, 2, 3, \ldots\}$: sequence of r.v.s

Definition: $Y_n \xrightarrow{a.s.} \mu$, or equivalently, $Y_n \longrightarrow \mu$ w.p.1, if

$$\mathbf{P}\left\{\lim_{n\to\infty} Y_n = \mu\right\} = 1$$

Question: Rewrite definition using sample paths:

- A sample path ω is "good" if: ____________________________
- A sample path ω is "bad" if: ______________________________

Question: What does almost sure convergence say?

5 Convergence Almost Surely: $Y_n \xrightarrow{a.s.} \mu$

$\{Y_n : n = 1, 2, 3, \ldots\}$: sequence of r.v.s

Y_n average of first n coinflips of fair coin

$$Y_n \xrightarrow{a.s.} \frac{1}{2} \qquad \text{if} \qquad \mathbf{P}\left\{\omega : \lim_{n\to\infty} Y_n(\omega) = \frac{1}{2}\right\} = 1.$$

$$\omega_1 = 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad \ldots$$

$$\omega_2 = 1 \quad 1 \quad 1 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad \ldots$$

FILL IN PIC:

For almost all sample paths, ω: $\quad Y_n(\omega) \to \frac{1}{2}$, as $n \to \infty$.

Question: Is every sample path ω good?

Question: How many "bad" sample paths are there? Finite? Infinite?

6 Convergence in Probability: $Y_n \xrightarrow{P} \mu$

$\{Y_n : n = 1, 2, 3, \ldots\}$: sequence of r.v.s

Definition: $Y_n \xrightarrow{P} \mu$ if $\forall \epsilon > 0$,

$$\lim_{n \to \infty} \mathbf{P}\left\{|Y_n - \mu| < \epsilon\right\} = 1$$

Question: Rewrite definition using sample paths:

Question: What is the sequence that we're taking the limit of?

Question: Suppose we instead want to look at the fraction of sample paths that are behaving badly at each n?

7 Convergence in Probability: $Y_n \xrightarrow{P} \mu$

$\{Y_n : n = 1, 2, 3, \ldots\}$: sequence of r.v.s
Y_n average of first n coinflips of fair coin

$$Y_n \xrightarrow{P} \frac{1}{2} \qquad \text{if} \qquad \forall \epsilon > 0, \ \lim_{n \to \infty} \mathbf{P} \left\{ \omega : \left| Y_n(\omega) - \frac{1}{2} \right| < \epsilon \right\} = 1$$

FILL IN PIC:

8 Comparison of two types of convergence

$$\text{a.s. convergence} \quad \stackrel{?}{\Longrightarrow} \quad \text{convergence in probability}$$

$$\text{a.s. convergence} \quad \stackrel{?}{\Longleftarrow} \quad \text{convergence in probability}$$

9 Laws of Large Numbers

Let $X_1, X_2, X_3, \ldots$ be i.i.d. r.v.s with mean μ.

Let $S_n = \sum_{i=1}^{n} X_i$. Let $Y_n = \frac{S_n}{n}$.

WLLN says: $Y_n \xrightarrow{P} \mu$ as $n \to \infty$

(Easy proof. See HW 2)

SLLN says: $Y_n \xrightarrow{a.s.} \mu$ as $n \to \infty$

(Hard proof. Omitted. But we use this result in our course.)

10 Commercial Break: Announcements

1. Pick up last of Probability Assessments.

2. Remember to bookmark class website:
   www.cs.cmu.edu/∼harchol/Perfclass/class.html

   - Go to **Announcements-and-Homeworks** link.
   - HW 1 due at start of class on Friday.
   - HW 2 being posted soon.
   - Look at syllabus too!

3. Mor has office hours TODAY: 5:30 p.m. in GHC 7207.

4. Please keep up with the reading for this class. Spend 3 hours after every class reading the relevant chapter. Read with pen in hand and cover up answers.

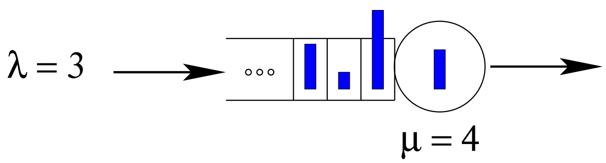
5. If you miss class, you can see me to photocopy my notes.

6. If you still don't have the textbook for this class, please come see me right after this class!!

7. Friday's class will be on z-transforms (see Chpt 25 of your queueing book, or Chpt 6 of the undergrad probability book). If you already know this topic from PnC, you can feel free to skip the class. However you may enjoy the review!

11 PART II: Two Kinds of Averages

Suppose you want to simulate a queue:



Let $N(v)$ = number jobs at time v.

GOAL: What is $\mathbf{E}[N]$? Two interpretations!

12 Definitions

$$\overline{N}^{\text{Time Avg}} = \lim_{t\to\infty} \frac{\int_0^t N(v)dv}{t} \qquad\qquad \overline{N}^{\text{Ensemble}} = \lim_{t\to\infty} \mathbf{E}\left[N(t)\right]$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$\overline{N}^{\text{Time Avg}}(\omega) = \lim_{t\to\infty} \frac{\int_0^t N(v,\omega)dv}{t} \qquad\qquad \overline{N}^{\text{Ensemble}} = \lim_{t\to\infty} \underbrace{\mathbf{E}\left[N(t,\omega)\right]}_{\text{over all } \omega}$$

13 Ergodic Theorem

(will discuss when get to Chpt 9 of book)

THM: For an ergodic system, the ensemble avg. exists, and

$$\overline{N}^{\text{Time Avg}} = \overline{N}^{\text{Ensemble}} \qquad \text{w.p. } 1$$

Question: What does this theorem say about almost all sample paths, ω?

Question: There's a key property of "ergodicity" that makes this hold. What do you think this is?

Question: When simulating, which average would you use?