1 Generating R.V.s for Simulation

UNIX gives you instances of Uniform$(0, 1)$.

How can you convert these to instances of some other distribution, X?

2 Inverse Transform Method for Continuous R.V.s

Given: Instance u of Uniform$(0, 1)$.

Want: Instance x of r.v. X: continuous, non-negative.

Assume we know: $F_X(x) = \mathbf{P}\{X \leq x\}$ and it is invertible.

3 Example of Inverse Transform for Continuous R.V.s

Question: How can we generate an instance of $X \sim \mathrm{Exp}(\lambda)$?

4 Inverse Transform Method for Discrete R.V.s

Given: Instance u of Uniform$(0, 1)$.

Want: Instance x of r.v. X: discrete

Assume Know: $F_X(x) = \mathbf{P}\{X \leq x\}$ and it is invertible

$$X = \begin{cases} x_0 & \text{with prob } p_0 \\ x_1 & \text{with prob } p_1 \\ \vdots & \\ x_k & \text{with prob } p_k \end{cases}.$$

5 Example of Inverse Transform for Discrete R.V.s

Generate instance x of X following a Geometric distribution:

$$Pr[X = k] = p^{k-1}(1-p).$$

6 Accept-Reject Method: High-level

Sometimes you don't know $F_X(x)$ or you know it but you can't invert it. Need a new method!

High-level idea behind the Accept-Reject Method:

- Want to generate instances of r.v. X.
- Instead generate instances of r.v. Y, over same range.
- Reject some values and Accept others.

7 Accept-Reject Method for Discrete R.V.s

Goal: Generate instance of X with p.m.f. $p_X(i)$.

Step 1: Find distribution Y with p.m.f. $p_Y(i)$ where

$$p_Y(i) > 0 \Leftrightarrow p_X(i) > 0$$

and where we already know how to generate Y.

Step 2: Let $c > 1$: smallest constant s.t.

$$\frac{p_X(i)}{p_Y(i)} \leq c, \qquad \forall i, \text{ s.t. } p_X(i) > 0 \ .$$

Step 3: Generate instance i of Y.

Step 4: With probability $\frac{p_X(i)}{cp_Y(i)}$, accept i, and return $X = i$.
Else, reject i and return to Step 3.

Question: What's the intuition behind step 4?

Question: Why did we need c here?

8 Analysis of Accept-Reject Method

Question: What's the fraction of time i is generated and accepted?

Question: What's the fraction of time any value is accepted?

Question: On average, how many values of Y are generated before one is accepted?

Question: What's the probability that X is set to i (as opposed to some other value)?

9 Accept-Reject Method for Continuous R.V.s

Goal: Generate X with p.d.f. $f_X(t)$.

Step 1: Find distribution Y with p.d.f. $f_Y(t)$ where

$$f_Y(t) > 0 \Leftrightarrow f_X(t) > 0$$

and where we already know how to generate Y.

Step 2: Let $c \geq 1$: smallest constant s.t.

$$\frac{f_X(t)}{f_Y(t)} \leq c, \qquad \forall t, \text{ s.t. } f_X(t) > 0 \; .$$

Step 3: Generate instance t of Y.

Step 4: With probability $\frac{f_X(t)}{cf_Y(t)}$, accept t, and return $X = t$.
Else, reject t and return to Step 3.

Question: Explain why again the expected number of iterations needed to get an instance of X is c.

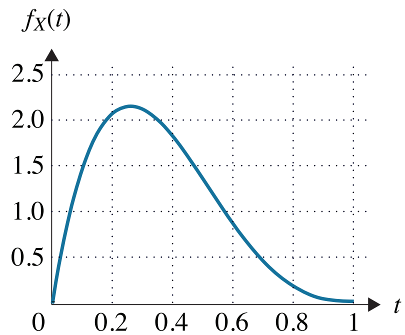
10 Generating a bounded continuous distribution

Goal: Generate r.v. X s.t.

$$f_X(t) = 20t(1-t)^3, \qquad 0 < t < 1 \ .$$

Question: Why can't we use inverse-transform method?

Question: Looking at $f_X(t)$ below, what's a good suggestion for Y?



Question: What's your algorithm?

Question: How many iterations on avg are needed to get an instance of X?

11 Generating a Normal distribution

Goal: Generate $N \sim \text{Normal}(0, 1)$

$$f_N(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}, \qquad -\infty < t < \infty \ .$$

DRAW PICTURE:

Question: Is there another r.v. Y that looks like this?

Question: Suppose we instead generate $X = |N|$. How do we convert X to N?

Question: What r.v. Y should we use to generate X?

12 Generating a Normal distribution, cont.

Question: What is c?

Question: What's our algorithm?

Question: What's the expected number of iterations?