Pop Quiz

Let $X \sim \text{Exp}(\lambda)$. Derive $\mathbf{E}\left[X^3\right]$.

Laplace Transform

Definition: Let X be a non-negative, continuous r.v. Then the Laplace transform of X is denoted by $\widetilde{X}(s)$ where:

$$\text{“Onion”} = \widetilde{X}(s) = \int_{t=0}^{\infty} e^{-st} f_X(t)dt = \underline{\hspace{3cm}}$$

Think of s as some constant (a place-holder), where $s \geq 0$.

Question: What is $\widetilde{X}(0)$?

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Question: Let $X \sim \text{Exp}(\lambda)$. What is $\widetilde{X}(s)$?

Question: Let $X = a$, where a is a constant. What is $\widetilde{X}(s)$?

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Think of s as some constant (a place-holder), where $s \geq 0$.

Question: Let $X \sim \text{Uniform}(a, b)$, where $a, b \geq 0$. What is $\widetilde{X}(s)$?

Question: How is the Laplace onion different from the z-transform onion?

Peeling the Onion

Theorem[Onion Peeling]: Let X: non-negative, continuous r.v. with p.d.f. $f_X(t)$, $t \geq 0$.

$$\begin{aligned}
\left.\widetilde{X}'(s)\right|_{s=0} &= -\mathbf{E}[X] \\
\left.\widetilde{X}''(s)\right|_{s=0} &= \mathbf{E}\left[X^2\right] \\
\left.\widetilde{X}'''(s)\right|_{s=0} &= -\mathbf{E}\left[X^3\right] \\
\left.\widetilde{X}''''(s)\right|_{s=0} &= \mathbf{E}\left[X^4\right] \\
&\vdots
\end{aligned}$$

If the above expressions are not defined at $s = 0$, then we can take the limit as $s \to 0$.

$X \sim \text{Exp}(\lambda)$. Use $\widetilde{X}(s)$ to get all moments of X.

Proving the Onion Peeling Theorem

Fill in the blanks below (write at least 5 terms):

$$e^{-sX} =$$

$$\widetilde{X}(s) = \mathbf{E}\left[e^{-sX}\right] =$$

$$\widetilde{X}'(s) =$$

$$\widetilde{X}'(0) =$$

$$\widetilde{X}''(s) =$$

$$\widetilde{X}''(0) =$$

$$\widetilde{X}'''(s) =$$

$$\widetilde{X}'''(0) =$$

Linearity theorem

<u>Theorem</u>: Let X and Y be continuous, non-negative, independent r.v.s. Let $Z = X + Y$. Then

$$\widetilde{Z}(s) = \widetilde{X}(s) \cdot \widetilde{Y}(s)$$

Proof:

Conditioning theorem

<u>Theorem</u>: Let X, A, and B be continuous r.v.s where

$$X = \begin{cases} A & \text{w.p. } p \\ B & \text{w.p. } 1 - p \end{cases}$$

Then

$$\widetilde{X}(s) = \underline{\hspace{6cm}}$$

Proof:

Random variables that depend on other r.v.s

Theorem: Let Y be a non-negative continuous r.v., and let X_Y be a continuous random variable that depends on Y. Then

$$\widetilde{X_Y}(s) = \int_{y=0}^{\infty} \widetilde{X_y}(s) \cdot f_Y(y) dy$$

Proof:

Example: A lightbulb can burn out at any time, independent of how long it has been running, so it makes sense to model its lifetime as an Exponential random variable.
Suppose that we know that this variety of lightbulbs has mean lifetime between 1/2 year and 1 year.

We model lightbulb's lifetime: $X_Y \sim \text{Exp}(Y)$ where $Y \sim$ ________________.

Derive $\widetilde{X_Y}(s)$.

Sum of random number of random variables

Let

$$S = X_1 + X_2 + \cdots + X_N$$

where
– N is a positive discrete integer-valued r.v.;
– X_i's are all continuous, i.i.d. r.v.s which are also independent of N;
– $X_i \sim X$

What is $\widetilde{S}(s)$?

Sum of Geometric number of Exponential random variables

Consider the sum of a Geometric(p) number of i.i.d. Exp(λ) random variables. How is this quantity distributed?

Fill out your cheat sheet

$$\widetilde{X}(s) \quad = \qquad\qquad\qquad\qquad \text{(Onion building)}$$

$$\mathbf{E}\left[X^k\right] \quad = \qquad\qquad\qquad\qquad \text{(Onion peeling)}$$

Distribution	$f_X(x)$	$\widetilde{X}(z)$	$\mathbf{E}[X]$
$\text{Unif}[a, b]$			
$\text{Exp}(\lambda)$			

Distribution	$\widetilde{S}(z)$
$S = X + Y, \quad X \perp Y$	
$S = \begin{cases} X & \text{with prob. } p \\ Y & \text{with prob. } 1-p \end{cases}$	
$S = \sum_{i=1}^{N} X_i$ $X_i \overset{\text{iid}}{\sim} X$ $N \perp (X_1, X_2, \dots)$	