1 Motivating example: Moments of Binomial(n,p)

$X \sim \text{Binomial}(n, p)$

What is $\mathbf{E}\left[X^3\right]$?

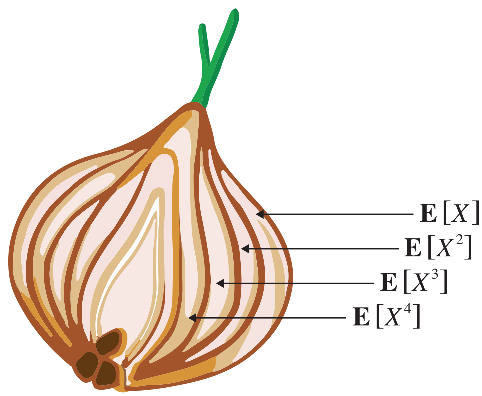
There must be a better way ...

2 z-Transform Definition

Definition: Let X be a discrete, non-negative, integer-valued r.v.
The **z-transform of X** is:

$$\widehat{X}(z) = \mathbf{E}\left[z^X\right] = \sum_{k=0}^{\infty} p_X(k) \cdot z^k$$

Assume z is a constant and $|z| \leq 1$.



Question: What is $\widehat{X}(1)$?

<u>**Outline for lecture**</u>

1. **How to build an onion**
2. **How to peel an onion**

3 Onion Building: Binomial

Definition: Let X be a discrete, non-negative, integer-valued r.v. The **z-transform of X** is:

$$\widehat{X}(z) = \mathbf{E}\left[z^X\right] = \sum_{k=0}^{\infty} p_X(k) \cdot z^k$$

$X \sim \text{Binomial}(n, p)$

Derive $\widehat{X}(z)$ in closed form.

4 Onion Building: Geometric

$X \sim \text{Geometric}(p)$

Derive $\widehat{X}(z)$ in closed form. (You will need to use the fact that $|z| \leq 1$).

5 Convergence of the z-transform

Theorem: $\widehat{X}(z)$ is bounded for any non-negative, discrete, integer-valued r.v. X, assuming $|z| \leq 1$.

PROOF:

Important: The definition of z-transform, $\mathbf{E}\left[z^X\right]$, can in theory apply to *any* r.v., not necessarily satisfying the above conditions. However it's not always obvious how to then prove that convergence holds.

6 Peeling the Onion

Theorem[Onion Peeling] Let X: non-negative, discrete, integer-valued r.v.

$$\begin{aligned}
\widehat{X}(z)\Big|_{z=1} &= 1 \\
\widehat{X}'(z)\Big|_{z=1} &= \mathbf{E}[X] \\
\widehat{X}''(z)\Big|_{z=1} &= \mathbf{E}[X(X-1)] \\
\widehat{X}'''(z)\Big|_{z=1} &= \mathbf{E}[X(X-1)(X-2)] \\
\widehat{X}''''(z)\Big|_{z=1} &= \mathbf{E}[X(X-1)(X-2)(X-3)] \\
&\vdots
\end{aligned}$$

If the above expressions are not defined at $z = 1$, then we can take the limit as $z \to 1$, sometimes with the aid of L'Hospital's rule.

Example: $\underline{X \sim \text{Binomial}(n, p)}$. Use $\widehat{X}(z)$ to get $\mathbf{E}[X]$ and $\mathbf{E}\left[X^2\right]$.

7 Proving the Onion Peeling Theorem

Fill in the blanks below (write at least 6 terms)

$$\widehat{X}(z) =$$

$$\widehat{X}'(z) =$$

$$\widehat{X}''(z) =$$

$$\widehat{X}'''(z) =$$

$$\widehat{X}'(1) =$$

$$\widehat{X}''(1) =$$

$$\widehat{X}'''(1) =$$

8 Another example

$\underline{X \sim \text{Poisson}(\lambda)}$

Derive $\mathbf{Var}(X)$ via transforms.

9 Addition theorem

Theorem: [**Adding z-transforms**] Let X and Y be non-negative, discrete r.v.s, where $X \perp Y$. Let

$$W = X + Y$$

Then:

$$\widehat{W}(z) = \widehat{X}(z) \cdot \widehat{Y}(z)$$

FILL IN PROOF:

Example: Let $X \sim \text{Bernoulli}(p)$. Let $Y \sim \text{Binomial}(n, p)$.

Question: What is $\widehat{X}(z)$?

Question: Use $\widehat{X}(z)$ to get $\widehat{Y}(z)$

10 Conditioning theorem

Let X, A, and B be non-negative, discrete r.v.s where:

$$X = \begin{cases} A & \text{w/prob. } \frac{1}{2} \\ B & \text{w/prob. } \frac{1}{2} \end{cases}$$

Question: Is X the same as $Y = \frac{1}{2}A + \frac{1}{2}B$?

Question: What is $\mathbf{E}[X]$?

Question: What is $\mathbf{E}\left[z^X\right]$?

<u>Theorem</u>: **Conditioning in z-transforms** Let X, A, and B be non-negative, discrete r.v.s where

$$X = \begin{cases} A & \text{w/prob. } p \\ B & \text{w/prob. } 1-p \end{cases}$$

Then:

$$\widehat{X}(z) = \underline{\hspace{6cm}}$$

11 Another example

Let X_1 and X_2 be i.i.d. r.v.s , both with the same distribution as X.

$$Y = \begin{cases} X_1 & \text{w/prob. } 0.5 \\ X_2 + 1 & \text{w/prob. } 0.5 \end{cases}$$

Question: What is $\widehat{Y}(z)$?

Question: Peel the onion to get $\mathbf{E}[Y]$

12 Sum of a random number of random variables

Let $X_1, X_2, X_3, \ldots$ be i.i.d. discrete random variables, where $X_i \sim X$.

Let N be a positive, integer-valued, discrete r.v., where $N \perp X_i$, $\forall i$.

Let

$$S = \sum_{i=1}^{N} X_i$$

Question: What is $\widehat{S}(z)$?

13 Fill out your cheat sheet

Distribution	$\hat{X}(z)$	$\mathbb{E}[X]$	$\mathrm{Var}(X)$
$\mathrm{Ber}(p)$			
$\mathrm{Bin}(n, p)$			
$\mathrm{Geo}(p)$			
$\mathrm{Pois}(\lambda)$			

Distribution	$\hat{S}(z)$
$S = X + Y$	
$S = \begin{cases} X & \text{with prob. } p \\ Y & \text{with prob. } 1-p \end{cases}$	
$S = \sum_{i=1}^{N} X_i$ $X_i \overset{\text{iid}}{\sim} X$ $N \perp (X_1, X_2, \dots)$	