Recall: Renewal Theory

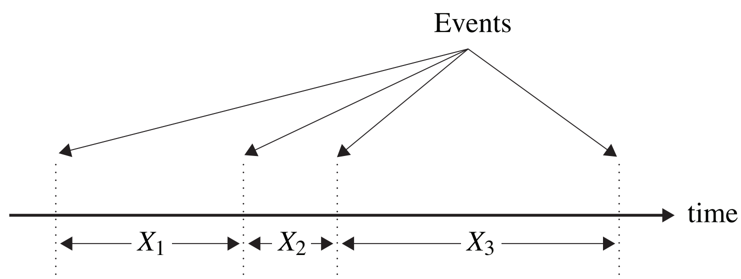


Figure 1: *A renewal process. The X_i's all have common distribution with mean* $\mathbf{E}[X]$.

Recall the Renewal Theorem (Thm 9.33): Given a renewal process.
Let $0 < \mathbf{E}[X] < \infty$ be the mean time between renewals (events).
Let $N(t)$ be the number of events by time t. Then, w.p.1:

$$\frac{N(t)}{t} \longrightarrow \underline{\hspace{3cm}} \text{ as } t \to \infty$$

Immediate consequence of Renewal Theorem for DTMCs

Theorem 9.28: For a positive recurrent, irreducible DTMC, w.p.1.,

$$p_j \equiv \lim_{t \to \infty} \frac{N_j(t)}{t} = \underline{\hspace{3cm}}$$

Renewal Reward Theory

During ith renewal, earn reward R_i.
R_i's are i.i.d. with mean $\mathbf{E}[R]$.
R_i may depend on X_i.

Renewal-Reward Theorem 23.4: Given a renewal-reward process, where $\mathbf{E}[X]$ is the mean time between renewals and $\mathbf{E}[R]$ is the mean reward earned during a renewal, where $0 \leq \mathbf{E}[R] < \infty$ and $0 < \mathbf{E}[X] < \infty$.
Let $R(t)$: total reward earned by time t. Then, with probability 1:

$$\frac{R(t)}{t} \longrightarrow \underline{\hspace{3cm}} \text{ as } t \to \infty$$

Proof of Renewal Reward Theorem

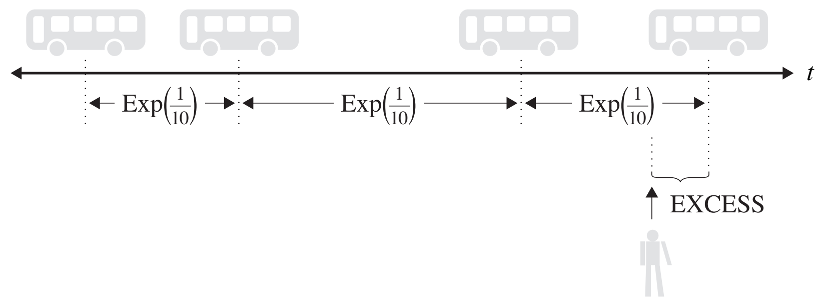
Renewal-Reward Theorem 23.4: Given a renewal-reward process, where $\mathbf{E}[X]$ is the mean time between renewals and $\mathbf{E}[R]$ is the mean reward earned during a renewal, where $0 \leq \mathbf{E}[R] < \infty$ and $0 < \mathbf{E}[X] < \infty$.
Let $R(t)$: total reward earned by time t. Then, with probability 1:

$$\frac{R(t)}{t} \longrightarrow \underline{\hspace{3cm}} \text{ as } t \to \infty$$

FILL IN PROOF:

GOAL for today: Closed-form analysis of M/G/1 (not M/PH/1).

Bus Question



A_i: time between bus i and bus $i+1$. A_i's are i.i.d. where $A_i \sim \text{Exp}(\frac{1}{10})$.

Question: If I arrive at random time, how long on average do I have to wait for bus?

(a) 5 minutes?

(b) 10 minutes?

(c) 20 minutes?

Question: What if $\mathbf{E}[A] = 10$, but A is not Exponentially distributed?

Definition: If A denotes the time between busses, then $\mathbf{A_e}$ is the **excess of A**, which is the time until the *next* bus, given I arrive at a random time.

M/G/1 queue

Draw queue where job sizes come from general distribution.
S denotes job size.

Consider only times when server is busy.

Question: I observe the system at a random time.
What do we call the remaining size for the job in service?

Tagged Job Analysis of M/G/1

We **"tag"** an arbitrary Poisson arrival and look at its experience.

T_Q : time in queue
N_Q : number in queue
N_Q^A : number in queue seen by arrival
S_i : service time of the i^{th} job in the queue
S : service time of a job, where $\mathbf{E}[S] = \frac{1}{\mu}$
S_e: excess of S. The remaining service time of the job in service, *given* that there's a job in service.

DERIVE $\mathbf{E}[T_Q]$:

Computing $\mathbf{E}[T_Q]$

$$\mathbf{E}[T_Q] = \frac{\rho}{1-\rho} \cdot \mathbf{E}[S_e]$$

Question: What does this say about M/M/1?

Question: What does this say about M/D/1?

Question: What does this say about M/E_k/1?

Question: What does this say about M/H_2/1?

GOAL: Need general formula for $\mathbf{E}[S_e]$ for any distribution S.

Commercial Break: Announcements

1. NEED REPLACEMENT GRADER for today. We are short 1 grader today (due to illness). Is anyone else available, or hungry for a free dinner?

2. HW 10 covers Chpt 23, specifically Renewal-Reward. Renewal-Reward problems have short solutions, but require cleverness. You'll need to define a reward process so that you can apply the formula:

$$\lim_{t \to \infty} \frac{R(t)}{t} = \frac{\mathbf{E}[R]}{\mathbf{E}[X]}.$$

   Note that there are 3 terms in this formula. In every problem, you will see that you actually know 2 of the terms and are being asked to find the 3rd term. The difficulty is figuring out which 2 terms you already know.

3. If you have not signed up for a 1-on-1 research meeting with me, please do so. I love to hear how queueing might fit into your research or life!

Recall: Renewal Reward Theorem

Renewal-Reward Theorem: Given a renewal-reward process:
Let $R(t)$: total reward earned by time t.
Let X denote the time between renewals.
If $0 \leq \mathbf{E}[R] < \infty$ and $0 < \mathbf{E}[X] < \infty$, then, with probability 1,

$$\lim_{t \to \infty} \frac{R(t)}{t} = \underline{\hspace{3cm}}$$

We will use this to get $\mathbf{E}[S_e]$!

The question is how to represent $\mathbf{E}[S_e]$ as a time-average reward rate.

Using Renewal Reward Theory to Derive $\mathbf{E}[\mathbf{S_e}]$

Question: Draw a picture of S_e as a function of time. Remember server is assumed to be always busy.

Question: Express $\mathbf{E}[S_e]$ as a time average.

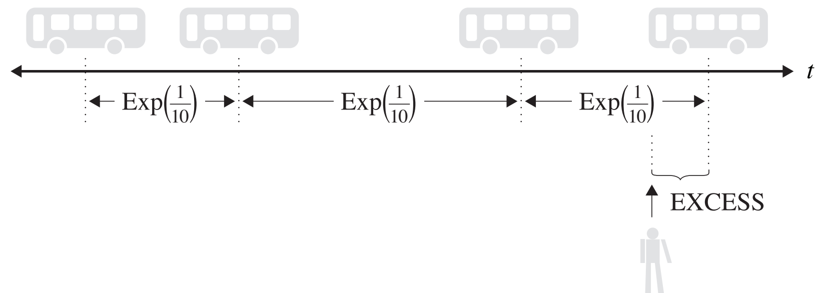
Question: Apply Renewal Reward to get $\mathbf{E}[S_e]$.

Pollaczek-Khinchin (P-K) Formula for $\mathbf{E}[T_Q]$

Question: What is $\mathbf{E}[T_Q]$?

Question: Rewrite $\mathbf{E}[T_Q]$ in terms of C_S^2.

Back to waiting for the bus ...



Assume times between buses are i.i.d. and are denoted by r.v. A.

Question: What is $\mathbf{E}[A_e]$?

Question: How high can $\mathbf{E}[A_e]$ be for general A?

Question: What is the expected time between buses **as observed by person arriving at random time?**

Question: What is the Inspection Paradox? Why does it happen?

Back to P-K formula. What we've learned about delay?

FILL IN P-K FORMULA:

Question: How does load ρ influence delay?

Question: How does C_S^2 influence delay?

Question: Company story: "Utilization is low, so why do I have high delay?"