Motivation

Last Lecture: Real-world job sizes are NOT Exponentially distributed. Inter-arrival times may also not be Exponentially distributed.

Today's Question: How can we represent non-Exponential distributions via Markov chains?

$$C_X^2 = \underline{\hspace{3cm}}$$

Question: Which distribution has $C_X^2 = 0$? ______________________

Question: What is C_X^2 for $X \sim \text{Exp}(\mu)$? ______________________

Representing Low Variance Distributions

Question: Suppose X has $C_X^2 < 1$ and $\mathbf{E}[X] = \frac{1}{\mu}$.

How can we represent X with Exponentials?

Defn: Erlang-k

Defn: Generalized-Erlang-k

The $M/E_2/1$ Queue

Representing High Variance Distributions

Question: Suppose X has $C_X^2 > 1$ and $\mathbf{E}[X] = \frac{1}{\mu}$.

How can we represent X with Exponentials?

Question: Does H_2 have IFR, DFR, CFR? No calculations. Just think!

The $M/H_2/1$ Queue

The Degenerate Hyperexponential

For some intuition for why the Hyperexponential is good at representing high variance distributions ... we turn to the Degenerate version:

Definition: The Degenerate Hyperexponential Distribution, H_2^*:
$X \sim H_2^*$ if

$$X \sim \begin{cases} \text{Exp}\,(p\mu) & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$$

Question: What is $\mathbf{E}\left[X\right]$?

Question: What is C_X^2?

Question: What is the range of C_X^2?

Question: Given any mean $\frac{1}{\mu}$ and $C^2 > 1$, can we always find an H_2^* match?

Can we have both?

Defn: A k-phase Coxian distributions has k Exponential phases organized as follows:

Coxian distributions have properties of both H_2^* and E_k!

Theorem [Cox 1950] Coxian distributions are *dense* in the class of non-negative distributions.

PH = Phase-type

Defn: A k-phase PH distribution has k Exponential phases, with arbitrary transitions between them.

Typical goal: Match > 3 moments of my distribution using PH with *few phases.*

[There is an entire research area devoted to this. See Taka Osogami papers]

An example of modeling with PH distributions

Analyzing infinite repeating CTMCs with multiple rows

We've now seen many CTMCs that:

- have multiple rows
- have an infinite repeating pattern

Here's one more example:

Question: How can we solve such infinite repeating Markov chains?

The Matrix-Analytic Methods (Very High-Level)

Recall: How did we solve M/M/1?

With multiple-row CTMCs:

Question: Where does R come from?

Three steps:

- Use ____________ to get the ____________ Q: ____________.
- Simplify Q using the repeating pattern of the CTMC.
- Substituting ____________ into the balance equation ____________ yields a matrix equation – from which R can be solved **numerically**.

Finally, use R and ____________ to solve for $\vec{\pi}_0$ and thus $\vec{\pi}_i$.