1 Outline for Today

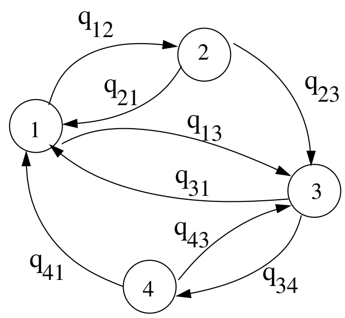
Motivation: Given enough computational effort, we can always solve balance equations for finite-state CTMCs. Infinite-state CTMCs are much harder. Infinite-state chains are particularly difficult when they are infinite in multiple dimensions.

Example:

Today:

1. Introduce new concept: "Reverse Chain."
2. Prove claims on Reverse Chain.
3. Explain Time-Reversibility in relation to the Reverse Chain.
4. Use Time-Reversibility and the Reverse Chain to prove Burke's Theorem.
5. Use Burke's Theorem to analyze infinite-state CTMCs which are infinite in multiple dimensions.

2 Vocabulary Reminder for CTMC



1. Rate of transitions from 1 to 2: ______________________

2. Rate of transitions from 1 to 2, given we're in 1: ______________________

3. Rate of transitions out of state 1: ______________________

4. Rate of transitions out of state 1, given we're in 1: ______________

5. $\mathbf{P}\{\text{Next go to 2} \mid \text{Currently in 1}\}$: ______________________

6. Express q_{13} in terms of ν_1: ______________________

7. Is the above chain time reversible? ______________________

8. Recall VIEW 1 versus VIEW 2

3 Reverse Process

Consider an ergodic CTMC in steady state. Imagine we are watching the CTMC as it transitions between states:

$$\cdots \longrightarrow 3 \longrightarrow 4 \longrightarrow 1 \longrightarrow 2 \longrightarrow 1 \longrightarrow 3 \longrightarrow 4 \longrightarrow 1 \longrightarrow \cdots$$

Now consider the **reverse process** for the CTMC. That is, we watch the CTMC, but we look *backward* in time (think of watching a movie being played in reverse):

$$\cdots \longleftarrow 3 \longleftarrow 4 \longleftarrow 1 \longleftarrow 2 \longleftarrow 1 \longleftarrow 3 \longleftarrow 4 \longleftarrow 1 \longleftarrow \cdots$$

Question: Is the reverse process also a CTMC? [Hint: Think about VIEW 1]

4 Properties of the Reverse Chain

All quantities associated with the reverse chain are tagged with an asterisk.

Question: Is $\nu_j = \nu_j^*$?

Question: Is $\pi_j = \pi_j^*$?

Question: Is $P_{ij} = P_{ij}^*$?

Question: Is $q_{ij} = q_{ij}^*$?

Recall: $\pi_i q_{ij}$ is the rate of transitions from i to j in the forwards chain.

Question: Is $\pi_i q_{ij} = \pi_i^* q_{ij}^*$?

Question: Is $\pi_i q_{ij} = \pi_j^* q_{ji}^*$?

5 More Properties of the Reverse Chain

Question: Based on what you just proved about the reverse chain, what is P^*_{ij}?

Question: Come up with an interpretation of the formula that you wrote above.

6 Time-Reversibility

<u>**Claim**</u>: If a CTMC is **time-reversible**, then the reverse chain is statistically identical to the forwards chain, meaning that they have the same CTMC.

FILL IN PROOF:

7 Commercial Break: Announcements

1. NO CLASS this Friday – enjoy a break!

2. Please turn in HW 7 at Mor's office on Friday at 2 p.m. (GHC 7207)

3. Mor's **office hours** today: **5:30 p.m. - 7 p.m.** Please come!

4. MIDTERM 2 on Dec 4th – 5-7 p.m. Put this on your calendar now. Will be very similar in style to Midterm 1 (very similar to homework problems).

5. No final in this class. Go home early!

6. Please keep up with the reading and go over homework solutions.

8 Burke's Theorem

Theorem:[Burke] Consider an M/M/1 system with arrival rate λ. Suppose the system starts in a steady state. Then the following are true:

1. The departure process is a Poisson Process with rate λ.
2. At each time t, the number of jobs in the system at time t is independent of the sequence of departure times prior to time t.

FILL IN ILLUSTRATION OF $N(t)$.

FILL IN PROOF OF BURKE:

9 Alternative Derivation of M/M/1 Departure Process

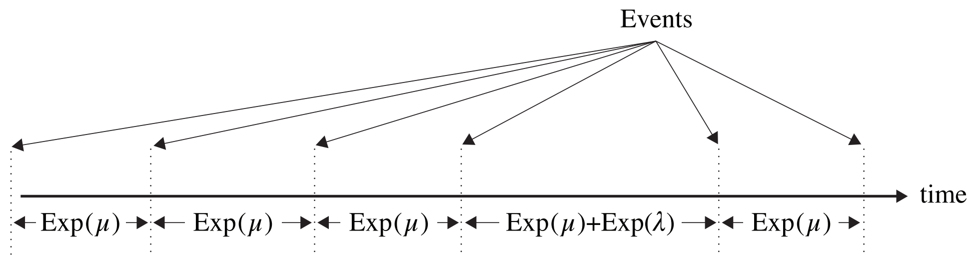
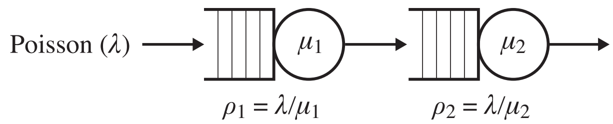


Figure 1: *Inter-departure times in M/M/1.*

10 Generalizations of Burke's Theorem

Question: Does Burke's Theorem work for an M/M/k?

11 Application: Tandem Servers



Let (n_1, n_2) represent the state: n_1 jobs at queue 1 and n_2 at queue 2.

Question: Write the balance equation for state (n_1, n_2), where $n_1, n_2 \geq 1$:

These balance equations look hard to solve!

Question: What does Burke's Theorem Part I tell us?

Question: What does Burke's Theorem Part II tell us?

12 General Acyclic Networks

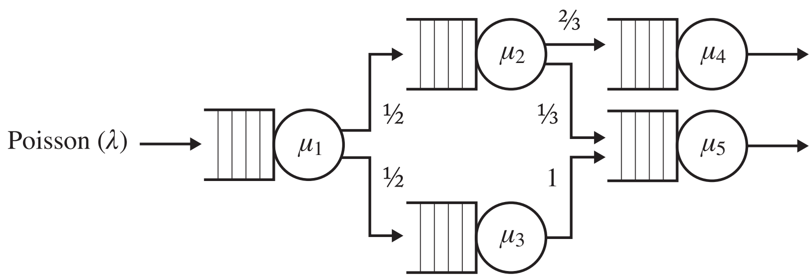


Figure 2: *An acyclic network of servers.*

N_1 = number of jobs at server 1. N_2 = number of jobs at server 2, etc.

Question: By Burke's theorem, what is $\mathbf{P}\{N_1 = n_1 \ \& \ N_2 = n_2 \ \& \ \dots \ \& \ N_k = n_k\}$?

Question: What is $\mathbf{P}\{N_1 = n_1\}$?