1 The M/M/k/k Loss System

The figure below shows the M/M/k/k loss system. The arrival process is a Poisson Process with average rate λ. The service time at each server is $\text{Exp}(\mu)$. Arrivals which find all servers busy are dropped.

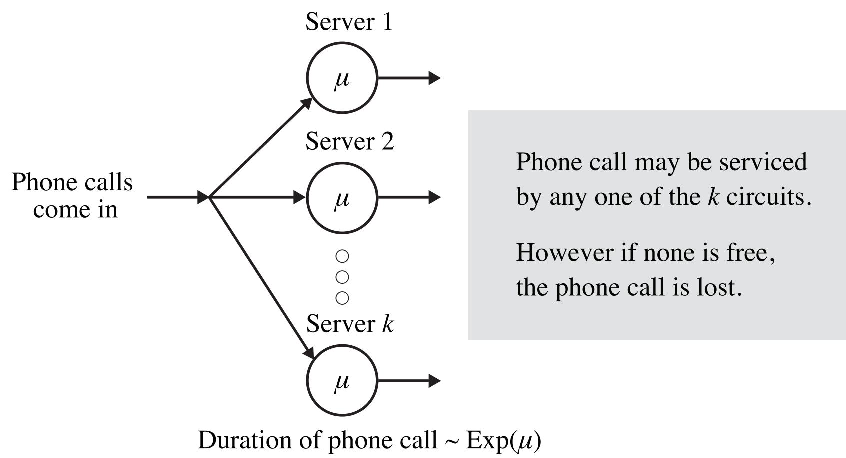

Figure 1: *M/M/k/k*

Our **goal** is to find the **blocking probability**, P_{block}, where

$$P_{block} = \mathbf{P}\left\{\text{Arrival is blocked/dropped}\right\}$$

1. Are we allowed to have $\lambda > \mu$?

2. Draw the appropriate CTMC.

3. Determine π_i and π_0.
(Observe that $\frac{\lambda}{\mu}$ is no longer related to the frac of time the server is busy.)

4. What is P_{block}? [Hint: PASTA] [Called “Erlang-B formula”]

5. There’s a more intuitive formulation of P_{block}.
[Hint: Think about the Poisson distribution]

6. What's the intuition behind your P_{block} formula?

2 M/M/k

In the M/M/k queueing system, jobs arrive according to a Poisson Process with average rate λ. The service time at each server is Exp(μ). Arrivals which find all servers busy are held in the queue.

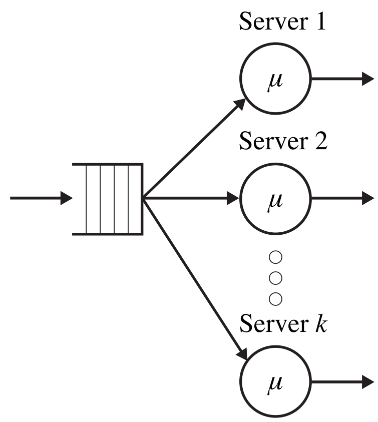

Figure 2: *M/M/k*

1. Draw the appropriate CTMC.

2. Determine π_i for the case where $i \leq k$ and the case where $i > k$.

3. Define the "system utilization" (a.k.a. system load) by:

$$\rho = \frac{\lambda}{k\mu}$$

What is the meaning of ρ?

4. Let R denote the expected number of busy servers. Derive R in 2 ways!

5. What's another interpretation for R?

6. **Back to the limiting probabilities**

Using $\rho = \frac{\lambda}{k\mu}$, we can write the limiting probabilities as:

$$\pi_i = \begin{cases} \frac{(k\rho)^i}{i!} \cdot \pi_0 & \text{if } i \leq k \\ \frac{\rho^i}{k!} \cdot k^k \cdot \pi_0 & \text{if } i > k \end{cases}$$

Question: What is π_0?

Answer:

$$\begin{aligned} \pi_0 + \sum_{i=1}^{k-1} \pi_i + \sum_{i=k}^{\infty} \pi_i &= 1 \\ \pi_0 \left[1 + \sum_{i=1}^{k-1} \frac{(k\rho)^i}{i!} + \sum_{i=k}^{\infty} \frac{\rho^i}{k!} k^k \right] &= 1 \\ \pi_0 \left[\sum_{i=0}^{k-1} \frac{(k\rho)^i}{i!} + \frac{k^k}{k!} \frac{\rho^k}{1-\rho} \right] &= 1 \\ \Rightarrow \pi_0 &= \left[\sum_{i=0}^{k-1} \frac{(k\rho)^i}{i!} + \frac{(k\rho)^k}{k!(1-\rho)} \right]^{-1} \end{aligned}$$

7. Let P_Q denote the probability that an arriving job has to queue. Prove that:

$$P_Q = \frac{(\rho k)^k}{k!} \cdot \frac{1}{1-\rho} \cdot \pi_0$$

You will need to use:

$$\pi_i = \begin{cases} \frac{(k\rho)^i}{i!} \cdot \pi_0 & \text{if } i \leq k \\ \\ \frac{\rho^i}{k!} \cdot k^k \cdot \pi_0 & \text{if } i > k \end{cases}$$

8. $\mathbf{E}[N_Q]$ is the expected number of jobs in the queue portion of the M/M/k.

Question: Derive $\mathbf{E}[N_Q]$

Answer:

$$\begin{aligned}
\mathbf{E}[N_Q]^{M/M/k} &= \sum_{i=k}^{\infty} \pi_i (i-k) \\
&= \pi_0 \sum_{i=k}^{\infty} \frac{\rho^i k^k}{k!} \cdot (i-k) \\
&= \pi_0 \frac{\rho^k k^k}{k!} \sum_{i=k}^{\infty} \rho^{i-k} \cdot (i-k) \\
&= \pi_0 \frac{\rho^k k^k}{k!} \sum_{i=0}^{\infty} \rho^{i} \cdot (i) \\
&= \pi_0 \frac{\rho^k k^k}{k!} \cdot \rho \cdot \frac{1}{(1-\rho)^2} \\
&= P_Q \cdot \frac{\rho}{1-\rho}
\end{aligned}$$

9. We proved that
$$\mathbf{E}\left[N_Q\right] = P_Q \cdot \frac{\rho}{1-\rho}.$$
BUT WHY??

10. Given $\mathbf{E}\left[N_Q\right]$, derive $\mathbf{E}\left[T_Q\right]$, $\mathbf{E}\left[T\right]$, and $\mathbf{E}\left[N\right]$.

11. What is $\mathbf{E}\left[N\right] - \mathbf{E}\left[N_Q\right]$? Does this make sense?

3 Commercial Break: Announcements

1. If you're missing anything, it's probably available at my office. Come to GHC 7207!
   Mor's Office Hours: 5:30 p.m. - 7:00 p.m. today. I will stay extra late if needed!

2. Make sure that you have carefully read all solutions given to you. This includes HW 6. Redoing HW problems is never a bad idea. I've been known to put homework problems on exams.

3. Fill in your index card now! Good things to include:

   - For all common distributions: p.d.f. or p.m.f.s, c.d.f., mean, variance, transform
   - Statement of important theorems

4. Please show up on time for the exam: 5 p.m., GHC 4307, Oct 9.

5. The exam will be fair. It will feel like homework problems from across our topics covered. Like the homework, the problems are not tricky but do require some insight, so get sleep. Computation will be minimal.

4 Comparison of Three Server Organizations

Below are 3 different server organizations, all with total arrival rate λ and total service rate $k\mu$.

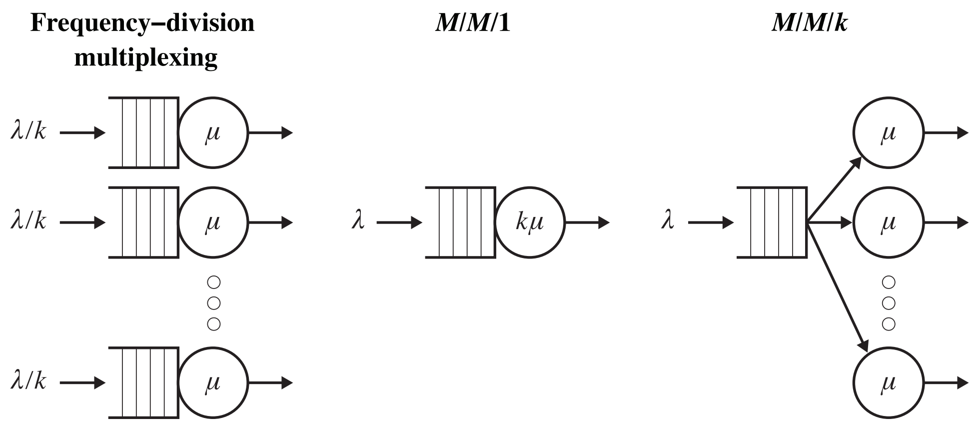

Figure 3: *Three architectures*

Question: How do these compare with respect to $\mathbf{E}[T]$?

5 Comparison of Three Server Organizations

Below are 3 different server organizations, all with total arrival rate λ and total service rate $k\mu$.

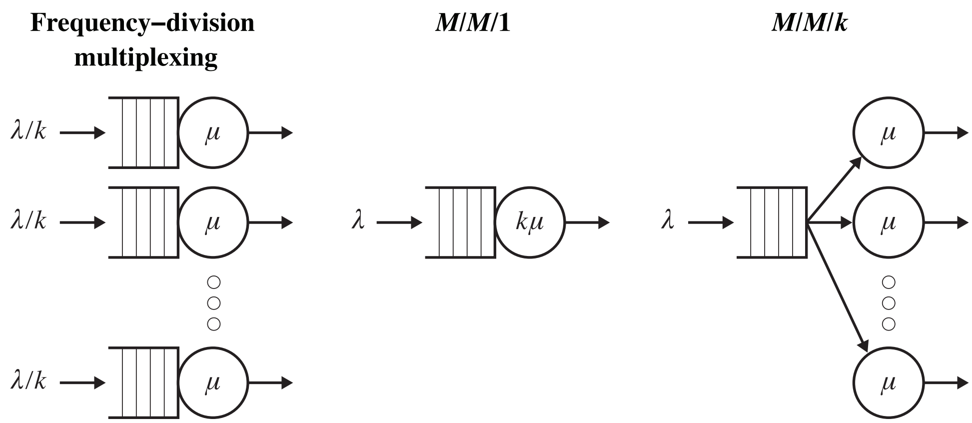

Figure 4: *Three architectures*

Question: How do these compare with respect to $\mathbf{E}[T_Q]$?