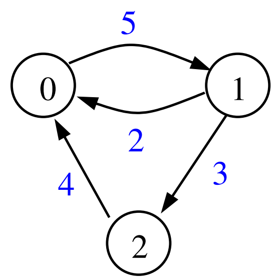
1 CTMC Review



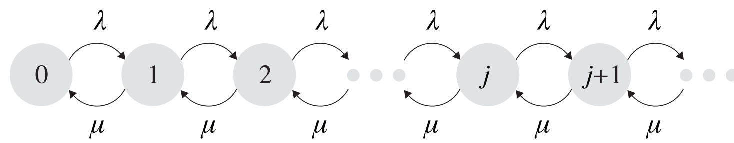
Question: How long do I stay in state 1 during each visit to state 1?

Question: When I finally leave state 1, what's the probability that I next go to state 2?

Question: What is the limiting distribution for this CTMC?

Question: Is the CTMC ergodic? How do I know the limiting distribution exists?

2 M/M/1 Review



Question: What is load, ρ?

Question: What are the balance equations?

Question: What would time-reversbility equations look like?

3 Time-Reversibility for CTMCs

Definition: A CTMC is **time-reversible** if, for all states i and j,

______________________ and ______________________

where π_i is the limiting probability of being in state i and q_{ij} is the rate of transitions from state i to state j given that the MC is in state i.

Theorem: Given an irreducible CTMC, suppose we can find x_i's such that

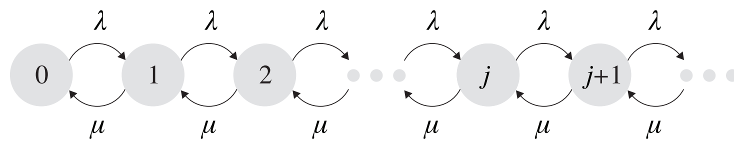
$$x_i q_{ij} = x_j q_{ji}, \quad \forall i, j \qquad \text{and} \qquad \sum_i x_i = 1$$

Then:

1. The x_i's are the limiting probabilities of the CTMC.
2. The CTMC is called "time-reversible."

PROVE IT!

4 M/M/1 Review



Question: What is π_i?

Question: How is N distributed?

Question: What is $\mathbf{E}[N]$? (memorize this)

Question: What is $\mathbf{E}[T]$? (memorize this)

5 Increasing Arrival and Service Rates Proportionally

Given an M/M/1 (with $\lambda < \mu$), suppose that we increase both λ and μ by a factor of k.

How are the following affected:

- utilization, ρ?
- throughput X?
- $\mathbf{E}[N]$?
- $\mathbf{E}[T]$?

"A transmission line k times as fast will accommodate k times as many packets at k times smaller average delay per packet." [Gallagher]

Question: Why was $\mathbf{E}[T]$ affected, but not $\mathbf{E}[N]$?

6 Statistical Multiplexing (SM) vs. Frequency-Division Multiplexing (FDM)

Suppose m independent Poisson packet streams, each with avg arrival rate $\frac{\lambda}{m}$ packets/sec, are transmitted over a communication line that can serve μ packets/sec.

SM: Merge the m streams into a single stream and transmit over the line.

FDM: Keep the m streams separated, and split the line into m lines, each with capacity of $\frac{\mu}{m}$ packets/sec.

Question: Which has lower $\mathbf{E}[T]$?

Question: Why does one do FDM?

7 Commercial Break: Announcements

1. I'm passing out index cards for Oct 9 exam! Will put extras in bins outside my office (GHC 7207). Exam starts at 5 p.m. Intended time 5-7. If you need a little more time at the end, you're welcome to it. I don't want anyone to feel time pressure. Goal is for everyone to do well. Problems are similar in style to the HW problems you've been doing all semester.

2. Highly recommend that you use your card for formulas that you are likely to forget. All the common distributions. All the means, variances, transforms of these distributions. We can't help you with these during the test.

3. Please feel free to take pictures of my lecture notes. Extra homework solutions are outside my door (GHC 7207).

4. HW 5 solutions going out today. If you don't know why your homework solution was marked wrong, please go to **Zhouzi's office hours: GHC 6003 right after this class: 3:30 p.m. - 5 p.m.**

5. HW 6 solutions will go out in class on Wednesday.

8 PASTA

Motivating Story: You're simulating some queueing system. You want to determine the fraction of time the system has n jobs. To do this, you ask each arrival to the system whether it sees n jobs or not. You then track the fraction of arrivals that witnessed n jobs upon arrival. Does this give you the right answer?

Formalizing the Story:
Assume ergodic CTMC, where "state" = number jobs in the system.
DRAW A SINGLE SAMPLE PATH WALKING ALONG THE CTMC:

p_n = long-run frac of time the system has n jobs = ________________
(averaged along ___________ sample path)

π_n = limiting probability that there are n jobs = ________________
(averaged along ___________ sample paths)

Question: What do we know about p_n versus π_n?

<u>Defn</u> : a_n = long-run fraction of arrivals that see n jobs
$\stackrel{\text{w.p.1}}{=}$ limiting probability that an arrival sees n jobs

Question: Is $a_n = p_n$?

9 PASTA, cont.

Let d_n = long-run fraction of departures that leave behind n jobs (or equivalently the limiting probability that a departure leaves behind n jobs).

Question: Is $a_n = d_n$?

Question: PASTA says that $a_n = p_n$ under certain circumstances. What are these?

Question: What does PASTA stand for?

Question: What does PASTA say about running simulations?

10 PASTA, cont.

PROVE PASTA!

$$
\begin{aligned}
p_n = \pi_n &= \lim_{t\to\infty} \mathbf{P}\left\{N(t) = n\right\} \\
a_n &= \lim_{t\to\infty} \mathbf{P}\left\{N(t) = n \mid \text{Arrival occurred just after time } t\right\}
\end{aligned}
$$

11 The M/M/k/k Loss System

The figure below shows the M/M/k/k loss system. The arrival process is a Poisson Process with average rate λ. The service time at each server is Exp(μ). Arrivals which find all servers busy are dropped.

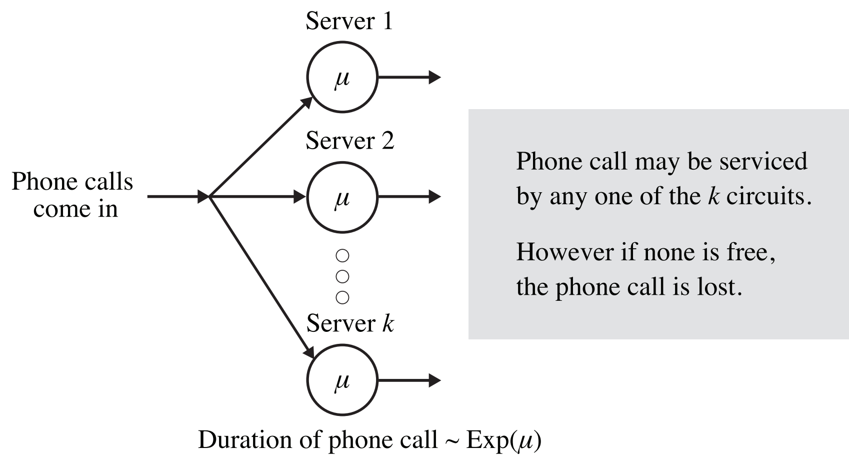


Figure 1: $M/M/k/k$

Our **goal** is to find the **blocking probability**, P_{block}, where

$$P_{block} = \mathbf{P}\left\{\text{Arrival is blocked/dropped}\right\}$$

1. Are we allowed to have $\lambda > \mu$?

2. Draw the appropriate CTMC.

3. Determine π_i and π_0.
   (Observe that $\frac{\lambda}{\mu}$ is no longer related to the frac of time the server is busy.)

4. What is P_{block}? [Hint: PASTA] [Called "Erlang-B formula"]

5. There's a more intuitive formulation of P_{block}.
   [Hint: Think about the Poisson distribution]

6. What's the intuition behind your P_{block} formula?