1 Continuous-time Markov Chains

Definition: A **DTMC** is a stochastic process $\{X_n : n = 0, 1, 2, \ldots\}$ where X_n denotes the state at discrete timestep n s.t., $\forall$ non-negative integers $i, j, i_{n-1}, i_{n-2}, \ldots,$

$$\mathbf{P}\{X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}, \ldots\}$$

$= \underline{\hspace{6cm}}$ (by the Markovian Property)

$= \underline{\hspace{6cm}}$ (by stationarity)

Definition: A **CTMC** is a stochastic process $\{X(t), t \geq 0\}$ where $X(t)$ denotes the state at time t s.t., $\forall s, t \geq 0$ and $\forall i.j, x(u)$,

$$\mathbf{P}\{X(t+s) = j \mid X(s) = i, X(u) = x(u), 0 \leq u \leq s\}$$

$= \underline{\hspace{6cm}}$ (by the Markovian Property)

$= \underline{\hspace{6cm}}$ (by stationarity)

$= \underline{\hspace{6cm}}$ (shorthand)

2 Understanding the Definition

Definition: Let τ_i denote the time until the CTMC leaves state i, given that the CTMC is currently in state i.

Question: What can we say about

$$\mathbf{P}\left\{\tau_i > s + t \mid \tau_i > s\right\}$$

3 Two EQUIVALENT VIEWS of CTMC

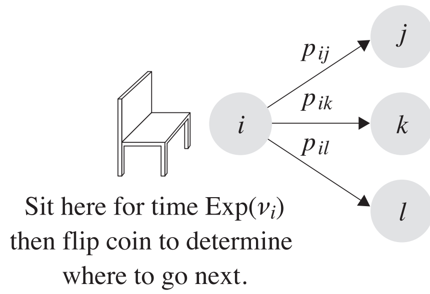

Figure 1: *VIEW 1 of a CTMC.*

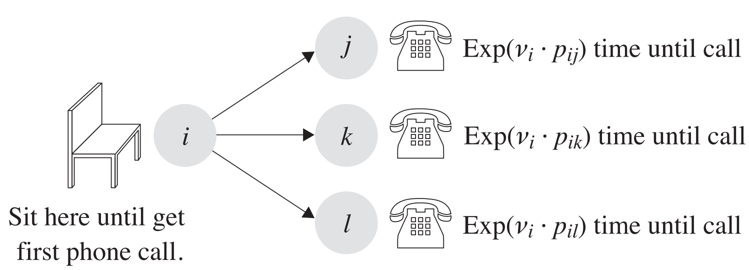

Figure 2: *VIEW 2 of a CTMC.*

4 Modeling a single-server queue

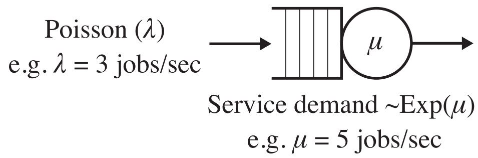

Figure 3: *A queue.*

Using VIEW 2, draw the CTMC model of this queue. Think only about events that cause a change of state.

OBSERVATIONS:

- λ and μ ____________________ probabilities.
- An "event" ______________ the state.
- Time to next arrival $\sim$ ____________________
- Time to next departure $\sim$ ____________________
- Time to leave state 1 $\sim$ ____________________
- $\mathbf{P}\{\text{When leave 1 next go to 2}\} =$ ____________________

5 Limiting Probabilities for CTMCs

Definition: Let $\pi_j = \lim_{t \to \infty} P_{ij}(t)$ = limiting probability of being in state j.

Question: How can we determine the π_j's?

Helping Question: If we had a DTMC, how would we determine π_j's?

Question: Can we model the CTMC via an equivalent DTMC?

- Think of time as discretized into δ-timesteps. Draw DTMC, where flip coin every δ timestep.

- Redraw DTMC using $o(\delta)$ notation.

6 Solving the Equivalent DTMC

- To solve DTMC, let's write "Balance Equations," which are equivalent to Stationary Eqns, but let us ignore self-loops

- Now divide all equations by δ and take the limit as $\delta \to 0$.

GENERAL MESSAGE: To solve CTMC, we pretend it's a __________ and then solve the ______________________________.

7 Converting general CTMCs to equivalent DTMCs

8 Summary Theorem for CTMCs

This follows directly from the "Summary Theorem" for DTMCs:

Theorem: [Summary Thm for CTMCs]
Given an irreducible CTMC.
Let q_{ij} denote the rate of transitions from state i to state j, given that the MC is in state i.
Let $\nu_i = \sum_j q_{ij}$. Suppose $\exists \pi_i$'s s.t. $\forall j$,

$$\pi_i \nu_i = \sum_j \pi_j q_{ji} \qquad \text{and} \qquad \sum_i \pi_i = 1$$

Then the π_i's are the limiting probabilities for the CTMC, and the CTMC is ergodic.

Define these terms:

- π_i
- q_{ij}
- $\pi_i q_{ij}$
- ν_i
- $\pi_i \nu_i$

9 M/M/1

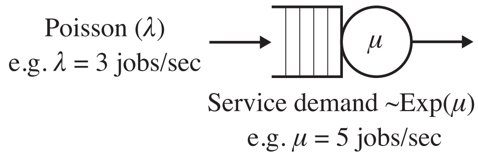

Figure 4: *M/M/1 queue*

Question: Why is this called an M/M/1?

Let N denote the number of jobs. Derive $\mathbf{E}[N]$ via these steps:

1. Draw the CTMC for the M/M/1

2. Write the balance equations for the CTMC.

3. Let $\rho = \frac{\lambda}{\mu}$. Make a guess for π_i in terms of ρ and π_0. Check your guess.

4. Solve for π_0. What does ρ represent?

5. Determine $\mathbf{E}[N]$.

6. Graph $\mathbf{E}[N]$ as a function of ρ.

PRACTICE PROBLEM:

Computer viruses appear according to a Poisson Process with avg rate λ viruses per hour.

Every hour we run a virus cleaner which removes all the viruses.

The *damage done by a virus* is directly proportional to the time that the virus is in the system (time from arrival until it's cleaned up).

Let D be a r.v. representing the total damage done by viruses in the first hour.

(a) What is $\mathbf{E}[D]$?

(b) What is $\widetilde{D}(s)$?