Where we're going: We've left DTMCs and are headed to CTMCs. But first we need a solid understanding of the Exponential distribution and the Poisson process.

1 Exponential Distribution: $\mathbf{X} \sim \mathbf{Exp}(\lambda)$

1. What is $f_X(x)$?

2. What is $\overline{F}_X(x)$?

3. What is $\mathbf{E}[X]$? What is $\mathbf{Var}(X)$?

2 Memoryless Property

Let X = Time until Verizon picks up my call.

Suppose that $X \sim \text{Exp}(\lambda)$. Let $s, t \geq 0$.

1. What is $\mathbf{P}\{X > t\}$?

2. What is $\mathbf{P}\{X > t + s \mid X > s\}$?

3. What is $\mathbf{E}[X \mid X > s]$?

4. What is $[X \mid X > s]$?

5. Based on this: what is $\mathbf{E}[X \mid X > s]$?

6. Based on this: what is $\mathbf{E}\left[X^2 \mid X > s\right]$?

3 More on Memoryless Property

The memoryless property says that

$$\mathbf{P}\{X > t + s \mid X > s\} = \rule{4cm}{0.4pt}$$

Question: What is a discrete distribution that has the memoryless property? Prove it!

Example of Memoryless Property:

Post Office has 2 clerks. Customers B and C arrive, each with service time $\sim$ Exp(λ). Customer A arrives (later), to see both B and C serving. Assuming that A's service time is also $\sim$ Exp(λ), what is $\mathbf{P}\{\text{A is last to leave}\}$?

4 Failure Rate Function

Definition: Let X: continuous r.v. The failure rate of X is:

$$r_X(t) = \frac{f_X(t)}{\overline{F_X}(t)}$$

Question: What does $r_X(t)$ mean?

Question: What is $r_X(t)$ when $X \sim \text{Exp}(\lambda)$?

Question: Let δ be very small. Suppose $X \sim \text{Exp}(\lambda)$ represents a disk's lifetime. What is the probability that the disk will fail in the next δ secs, given it has lasted t secs so far?

5 Yet another view of the Exponential distribution

The Exponential distrib is the continuous counterpart to the Geometric.
BUT HOW?

6 Review: Exponential Distribution

$X \sim \text{Exp}(\lambda)$ is the <u>time</u> until a coin with probability ___________

comes up heads, given that the coin is flipped every ___________ time.

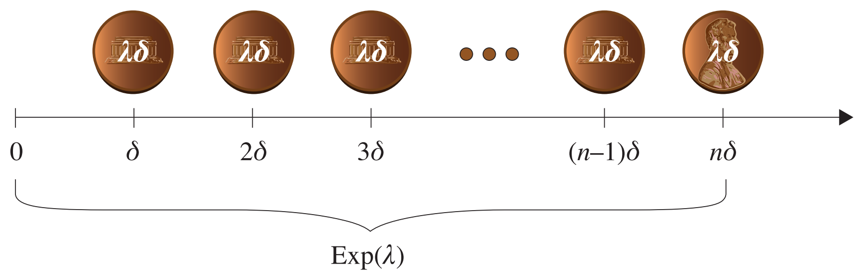

Figure 1: *Illustration of Exp(λ) distribution.*

7 A useful definition: $\mathbf{o}(\delta)$

<u>Definition</u>:

$$f = o(\delta) \quad \text{if} \quad \lim_{\delta \to 0} \frac{f}{\delta} = 0$$

FILL IN EXAMPLES:

8 Commecial Break: Announcements

1. HW 4 being returned. Avg was 87%. Key to success on homeworks: Start early, so you can hit office hours with a good understanding of where you're stuck.

2. Please review all HW solutions, not just problems that you got right.

3. Midterm: Oct 9th, 5-7 p.m. in usual classroom. You can bring one 3x5 index card with writing on both sides. I recommend a chart with all common distributions: p.m.f. or p.d.f, mean, variance, Laplace or z-transform, etc.

4. If you are missing HW solutions, the extras are in the bins outside my office door (GHC 7207). If you missed a class, you can take a picture of my notes. Find me in my office!

5. **Zhouzi has office hours today in GHC 6003, immediately after class!**

9 Which Exponential happens first

Theorem: Let $X_1 \sim \text{Exp}(\lambda_1)$ and $X_2 \sim \text{Exp}(\lambda_2)$ and $X_1 \perp X_2$. Then

$$\mathbf{P}\{X_1 < X_2\} = \underline{\hspace{5cm}}$$

Prove without integrals and without conditioning.
Use δ-steps which are more intuitive.

10 Min of two Exponentials

Theorem: Let $X_1 \sim \text{Exp}(\lambda_1)$ and $X_2 \sim \text{Exp}(\lambda_2)$ and $X_1 \perp X_2$.

Let $Y = \min(X_1, X_2)$. Then $Y \sim$ ________________
PROVE IT!

11 Towards Poisson Process

The Poisson Process is the most widely used model for outside arrivals into a system, for 2 reasons: (1) It represents the limiting process when many independent users are merged; (2) It is analytically tractable.

"Arrival Process" or "Event Sequence":

Independent Increments:

Stationary Increments:

Question: If $X \sim \text{Poisson}(\lambda)$, then $p_X(i) =$ ____________________.

12 Definition 1 of Poisson Process

A **Poisson process with rate** λ is a sequence of events such that:

1. $N(0) = 0$
2. The process has independent increments.
3. The process has stationary increments.
4. $N(t) \sim$ ____________________, $\quad \forall t$

Question: Why is λ called the "rate" of the process?

Question: What is $\mathbf{P}\left\{N(t+s) - N(s) = n\right\}$?

Question: What does it mean for a process to have both stationary and independent increments?

13 Definition 2 of Poisson Process

A **Poisson process with rate** λ is a sequence of events such that

the interarrival times are i.i.d. ____________________ variables.

Question: Which definition do we use for simulating a Poisson Process?