14 Little's Law – more detailed proof

In the Little's Law Proof, we have:

$$\sum_{j=1}^{C(t)} T_i \leq \int_0^t N(s)ds \leq \sum_{j=1}^{A(t)} T_i.$$

We then divide each side by t, yielding:

$$\frac{\sum_{j=1}^{C(t)} T_i}{t} \leq \frac{\int_0^t N(s)ds}{t} \leq \frac{\sum_{j=1}^{A(t)} T_i}{t}.$$

It is easy to see that the center term converges to $\overline{N}$ and RHS converges to $\overline{T} \cdot \lambda$. What students are less sure about is why the LHS converges to $\overline{T} \cdot \lambda$. We now show that this holds. In fact, in showing this, we don't need to use the fact that $X = \lambda$, so we can remove that from our theorem statement and simply say that all the limits must exist.

Claim: $\lim_{t\to\infty} \frac{\sum_{j=1}^{C(t)} T_i}{t} \geq \lambda \overline{T}$.

Proof:

Let T_j denote the response time of the jth job. Let t_j^d denote the departure time of the jth job and t_j^a denote its arrival time.

Choose an arbitrary $\epsilon > 0$ (can be arbitrarily small).

Then $\exists n_0$ s.t. $T_j \leq \epsilon \cdot t_j^a$.

But this tells us that

$$t_j^d = t_j^a + T_j \leq (1+\epsilon) t_j^a, \ \forall j \geq n_0.$$

Now, the set of jobs that have completed by time t is $\{j : t_j^d \leq t\}$, but from the above, we know that:

$$\{j : t_j^d \leq t\} \supset \{j : j \geq n_0 \ , \ (1+\epsilon) t_j^a \leq t\} = \{j : j \geq n_0 \ , \ t_j^a \leq \frac{t}{1+\epsilon}\}.$$

Hence

$$\sum_{j=1}^{C(t)} T_j \geq \sum_{j=n_0}^{A\left(\frac{t}{1+\epsilon}\right)} T_j = \sum_{j=1}^{A\left(\frac{t}{1+\epsilon}\right)} T_j - \sum_{j=1}^{n_0-1} T_j.$$

We now divide each term by t and take the limit as $t \to \infty$.

The last term is a finite sum, so it disappears when divided by t.

We now consider the remaining term, $\sum_{j=1}^{A\left(\frac{t}{1+\epsilon}\right)} T_j$, divide by t and take the limit:

$$\begin{aligned}
\lim_{t\to\infty} \frac{\sum_{j=1}^{A\left(\frac{t}{1+\epsilon}\right)} T_j}{t} &= \lim_{t\to\infty} \frac{\sum_{j=1}^{A\left(\frac{t}{1+\epsilon}\right)} T_j}{A\left(\frac{t}{1+\epsilon}\right)} \cdot \frac{A\left(\frac{t}{1+\epsilon}\right)}{t} \\
&= \overline{T} \cdot \lim_{t\to\infty} \frac{A\left(\frac{t}{1+\epsilon}\right)}{t} \\
&= \overline{T} \cdot \frac{\lambda}{1+\epsilon}
\end{aligned}$$

Hence we have shown that:

$$\sum_{j=1}^{C(t)} T_j \geq \overline{T} \cdot \frac{\lambda}{1+\epsilon}$$

Since the above holds for all ϵ, we can conclude that:

$$\sum_{j=1}^{C(t)} T_j \geq \overline{T} \cdot \lambda.$$