Homework #4: More DTMCs


Instructions: Feel free to collaborate with other students, but you should write up your own solutions. It is good form to list the names of people with whom you collaborate.

1 Problems: 9.7, 9.17a, 9.18, 10.4, 10.7, 25.1, 25.2

Some of the problems are not in the book, so I’ve included them below. Problem 10.7 is the challenge problem that I gave you earlier in class. I’ve included a bunch of helping steps below to make it easier. (Note: 9.17a will make more sense after we cover Thm 9.6 in the first 15 minutes of class on Monday.)

Exercise 9.17(a): [Easy Extensions to Theorems in Book] In the case of an irreducible, periodic, finite-state chain, Section 9.8 (not covered in class) tells us that the stationary distribution exists, even though the limiting distribution does not exist. Using the above fact as given, edit the proof of Theorem 9.6 to prove the following: For an irreducible, periodic, finite-state DTMC,

\[ m_{jj} = \frac{1}{\pi_j} \]

where now \( \pi_j \) is the stationary probability of being in state \( j \).

Exercise 9.18: [Stationary but not Limiting] We’ve seen several examples of finite-state DTMCs for which the stationary distribution exists, but the limiting distribution does not. Provide an example of an infinite-state irreducible DTMC for which there exists a unique stationary distribution, however the limiting distribution does not exist. Solve for the stationary distribution of your chain.

(SEE BACK PAGE!)
Exercise 10.7: [Processor With Failures]

Consider the DTMC shown in Figure 1.

This kind of chain is often used to model a processor with failures. The chain tracks the number of jobs in the system. At any time step, either the number of jobs increases by 1 (with probability $p$), or decreases by 1 (with probability $q$), or a processor failure occurs (with probability $r$), where $p + q + r = 1$. In the case of a processor failure, all jobs in the system are lost. Derive the limiting probability, $\pi_i$, of there being $i$ jobs in the system. Follow the steps below:

STEP 0: First try to do this without transforms, just by manipulating the stationary equations. Try it for an hour. After you give up, you’ll appreciate the approach in this problem.

STEP 1: Write the stationary equation for state 0. Use this to get an equation for $\pi_1$ in terms of $\pi_0$.

STEP 2: Write the stationary equations for state $i \geq 1$.

STEP 3: Let $\hat{\Pi}(z) = \sum_{i=0}^{\infty} \pi_i z^i$. Derive an expression for $\hat{\Pi}(z)$ in terms of $\pi_0$.

STEP 4: Rewrite $\hat{\Pi}(z)$ so that its denominator can be written in the form of $(1 - \frac{z}{r_1}) (1 - \frac{z}{r_2})$, where $r_1$ and $r_2$ are roots that you specify, where $r_1 < r_2$.

STEP 5: Determine $\pi_0$. Use the fact that $\hat{\Pi}(z)$ is bounded for all $0 \leq z \leq 1$ and the fact that $0 \leq r_1 < 1$.

STEP 6: Apply partial fraction decomposition to $\hat{\Pi}(z)$.

STEP 7: Rewrite $\hat{\Pi}(z)$ as a geometric series.

STEP 8: Match coefficients to get the $\pi_i$’s.

STEP 9: Verify that your solution for $\pi_i$ satisfies the stationary equations.