1. Example of Solving Finite-State CTMC

1. What is the throughput, $X$?
2. What is the throughput, assuming $N \to \infty$?
3. What is $E[T_{cpu}]$?
4. What is $\rho_{diskA}$?
2 Outline for Today

Finite-state CTMCs are always solvable, given enough computational effort. Infinite-state CTMCs are much harder. Infinite-state chains are particularly difficult when they are infinite in multiple dimensions.

1. Introduce new concept: “Reverse Chain.”

2. Prove claims on Reverse Chain.


4. Use Time-Reversibility and the Reverse Chain to prove Burke’s Theorem.

5. Use Burke’s Theorem to analyze infinite-state CTMCs which are infinite in multiple dimensions.
3 Reverse Process

Consider an ergodic CTMC in steady state. Imagine we are watching the CTMC as it transitions between states:

\[ \cdots \rightarrow 3 \rightarrow 5 \rightarrow 1 \rightarrow 2 \rightarrow 1 \rightarrow 3 \rightarrow 4 \rightarrow 1 \rightarrow \cdots \]

Now consider the reverse process for the CTMC. That is, we watch the CTMC, but we look \textit{backward} in time (think of watching a movie being played in reverse):

\[ \cdots \leftarrow 3 \leftarrow 5 \leftarrow 1 \leftarrow 2 \leftarrow 1 \leftarrow 3 \leftarrow 4 \leftarrow 1 \leftarrow \cdots \]

**Question:** Is the reverse process also a CTMC? [Hint: Think about VIEW 1]
4 Properties of the Reverse Chain

Assume all quantities associated with the reverse chain are tagged with an asterisk.

Question: Is $\nu_j = \nu_j^*$?

Question: Is $\pi_j = \pi_j^*$?

Question: Is $P_{ij} = P_{ij}^*$?

Question: Is $q_{ij} = q_{ij}^*$?

Note that $\pi_i q_{ij}$ represents the rate of transitions from $i$ to $j$ in the forwards chain.

Question: Is $\pi_i q_{ij} = \pi_i^* q_{ij}^*$?

Question: Is $\pi_i q_{ij} = \pi_j^* q_{ji}^*$?
5 More Properties of the Reverse Chain

Question: Based on what you just proved about the reverse chain, what is $P_{ij}^*$?

Question: Come up with an interpretation of the formula that you wrote above.
6 Time-Reversibility

Claim: If a CTMC is time-reversible, then the reverse chain is statistically identical to the forwards chain, meaning that they have the same CTMC.

FILL IN PROOF:
7  Burke’s Theorem

**Theorem 1 (Burke)** Consider an $M/M/1$ system with arrival rate $\lambda$. Suppose the system starts in a steady state. Then the following are true:

1. The departure process is Poisson($\lambda$).

2. At each time $t$, the number of jobs in the system at time $t$ is independent of the sequence of departure times prior to time $t$.

FILL IN PROOF:
Alternative Derivation of M/M/1 Dept. Process

Figure 1: Inter-departure times in M/M/1.
9 The algebra ...

\[ P \{ T > x \} = \rho e^{-\mu x} + (1 - \rho) \left( \int_{t=0}^{x} e^{-\mu(x-t)} \lambda e^{-\lambda t} dt + \int_{t=x}^{\infty} 1 \cdot \lambda e^{-\lambda t} dt \right) \]

\[ = \rho e^{-\mu x} + (1 - \rho) e^{-\mu x} \lambda \int_{t=0}^{x} e^{(\mu-\lambda)t} dt + (1 - \rho) e^{-\lambda x} \]

\[ = e^{-\mu x} \left( \rho + (1 - \rho) \lambda \frac{e^{(\mu-\lambda)x} - 1}{\mu - \lambda} + (1 - \rho) e^{(\mu-\lambda)x} \right) \]

\[ = e^{-\mu x} \left( \rho + \frac{(1 - \rho)\lambda (e^{(\mu-\lambda)x} - 1)}{\mu - \lambda} + (1 - \rho) e^{(\mu-\lambda)x} \right) \]

\[ = e^{-\mu x} \left( \rho + \frac{(1 - \rho)\lambda (e^{(\mu-\lambda)x} - 1)}{1 - \rho} + (1 - \rho) e^{(\mu-\lambda)x} \right) \]

\[ = e^{-\mu x} \left( \rho + \rho e^{(\mu-\lambda)x} - \rho + (1 - \rho) e^{(\mu-\lambda)x} \right) \]

\[ = e^{-\mu x} \left( e^{(\mu-\lambda)x} \right) \]

\[ = e^{-\lambda x}. \]

10 Generalizations of Burke’s Theorem

Question: Does Burke’s Theorem generalize to an M/M/k?

9
11 Application: Tandem Servers

Let \((n_1, n_2)\) represent the state of \(n_1\) jobs at queue 1 and \(n_2\) jobs at queue 2.

**Question:** Write the balance equation for state \((n_1, n_2)\):

These balance equations look hard to solve!

**Question:** What does Burke’s Theorem Part I tell us?

**Question:** What does Burke’s Theorem Part II tell us?
Let $N_1$ denote the number of jobs at server 1, $N_2$ denote the number of jobs at server 2, etc.

**Question:** By Burke’s theorem, what is $P\{N_1 = n_1 \& N_2 = n_2 \& \ldots \& N_k = n_k\}$?

**Question:** What is $P\{N_1 = n_1\}$?