1 $\Delta + 1$ coloring via MIS

Given a graph $G$ with $\Delta$-MIS. Color each vertex such that no edge connects two nodes of same color.

1.1 Solution

For $G = \langle N, E \rangle$, where $N = \{v_1, v_2, ..., v_n\}$, construct a new graph consisted of $n$ cliques,

$$G' = \bigcup_{i \leq n} C_i, E'$$

where

$$C_i = \{u_{i,1}, ..., u_{i,\Delta+1}\}$$

$$E' = \bigcup_{i \leq \Delta + 1} \{(u_{p,i}, u_{q,i}) : (v_p, v_q) \in E \} \cup \bigcup_{i \leq n} \{(u_{i,p}, u_{i,q}) : 1 \leq p < q \leq \Delta + 1\}$$

Notice that we split each node $v_i$ into $\Delta + 1$ connected nodes in new graph.

Now, we claim that $G'$ is a $\Delta + 1$-coloring graph.

Proof

At the first step, we prove that the size of MIS in $G'$ is exact $n$ by showing that for any clique $C_i$, exact one node will be selected into MIS. On the one hand, we can not put two nodes staring a common clique in MIS. On the other hand, any clique $C_i$ connects to at most $\Delta$ other cliques, each of which has at most one vertex that could be picked in MIS. That means at most $\Delta$ nodes in $C_i$ are adjacent to the nodes in other cliques that could be selected in MIS. Hence, at least one node should be selected in MIS for every clique.

Actually, if we mark each vertex $u_{i,j}$ with color $j$ and dye $v_i$ in the original graph $G$ with the color of the only selected vertex in $C_i$, which is picked in the MIS of $G'$, it would be a $\Delta + 1$ coloring profile. According to the property of MIS, no two nodes in MIS of $G'$ share a common edge. I.e., no two connected nodes in $G$ have the same color.

$\blacksquare$
1.2 Algorithm

From last lecture, $O(\log n)$ rounds is needed to find a valid MIS, where the message from one node to the other is just a generated random integer. Now what we need to do is running a similar algorithm on a new graph. This time, each network node $v_i$ simulates $\Delta + 1$ virtual nodes. Intuitively, $\Delta + 1$ generated random integers should be sent in each round.

Actually, each node should only send the maximum random integer in its own clique and the index of that node, because this integer is the only candidate that can participate a 'surviving' game, where a node with maximum integer wins. Next, each node should reply for each of his neighbor whether the received integer is larger than its real adjacent node’s. Finally, each node knows whether its candidate is a biggest and then broadcasts this information to help others update the graph. By this trick, the round complexity keeps the same.

2 Graph Shatter

2.1 Locality in graph problems

LOCAL model synchronous routes per round, send one (unbounded sized) message to each neighbor. One observation is that after $k$ rounds of the LOCAL model, each node knows its $k$-neighbors.

From last lecture, MIS can be computed in $O(\log n)$ rounds. In this section, we will attempt to solve it in $O(\log \Delta \sqrt{\log n})$ rounds, which is interesting only if $\Delta < 2^{\sqrt{\log n}}$.

2.2 Intuition

After running a few ‘Ruby’, the structure is becoming better. One perspective of a good structure is that the upper bound of all shortest paths is relatively low. This is because we can 'know’ everything after several rounds of communication. Hence, what we attempt to do is to halve $\Delta$ in $O(\sqrt{\log n})$ rounds such that after each halving, high degree nodes form a subgraph with good structure.

2.3 Algorithm

We run the following algorithm $O(\log \Delta)$ times.

Algorithm 1 Halving $\Delta$

1: for $c \sqrt{\log n}$ stages do
2: each node marks itself w.p. $\frac{1}{\Delta + 1}$
3: marked nodes join MIS if no neighbor is marked
4: remove neighbors of MIS nodes
5: end for
2.4 Analysis

Call a node red if its degree is at least $\Delta/2$. Set $\mathcal{H}$ as the graph induced by red nodes. What we need to show is that connected components of $\mathcal{H}$ have (mark) diameter at most $5\sqrt{\log n}$. Let’s look at following two lemmas.

Lemma 1 For each node $v$, w.p. $c' = \Omega(1)$, node $v$ will be removed at the end of this stage.

Proof For any node $v$, mark its neighbors as $u_1, u_2, \ldots, u_d$. Let $j^*$ be the minimum index of $v$’s neighbors that get marked. Hence,

$$P(\exists j^*) = 1 - \left(1 - \frac{1}{\Delta + 1}\right)^d \geq 1 - e^{-\frac{d}{\Delta + 1}} \geq 1 - \frac{1}{\sqrt{e}} = \Theta(1)$$

where $d \geq \frac{\Delta}{2}$.

Furthermore,

$$P(j^* \in MIS) = \left(1 - \frac{1}{\Delta + 1}\right)^{\Gamma(u_{j^*}) \setminus \{u_1, \ldots, u_{j^* - 1}\}} \geq \left(1 - \frac{1}{\Delta + 1}\right)^\Delta = \Theta(1)$$

Hence, node $v$ will be removed with constant probability.

For all $v$ and $v'$ at distance $5\sqrt{\log n} - 1$, let $\mathcal{P}$ be the set of all paths between $v$ and $v'$.

Lemma 2 There is $Q \subseteq \mathcal{P}$ that is 5 independent and $|Q| \geq \sqrt{\log n}$.

Proof We mark each node in the path with numbers $1, 2, \ldots$. Actually, we can construct $Q$ by choosing nodes $q_1, q_2, \ldots, q_k$ one by one: let $q_1 = v$; for any $i > 1$, we set $q_i$ as the node with max index that has distance 5 from $q_{i-1}$.

Obviously, $k$ is at least $\sqrt{\log n}$.

Suppose $q_i$ and $q_j$ are not 5 independent where $i < j$. We can definitely find another node $u$ that has higher number then $q_j$ and can be reached by $q_i$ in 5 steps through this path. Thus, we should have set $q_{i+1}$ as $u$, which makes a contradiction.

By these two lemmas, we can show that for each $P \in \mathcal{P}$, at least one node of $P$ is knocked out.

Let the event $Q : k$ be ’$Q$ stays for $k$ stages’. Based on first lemma,

$$P(Q : 1) \leq \exp\left(-\Theta\left(\sqrt{\log n}\right)\right)$$
Since it is 5 independent,

\[ P \left( Q : c \sqrt{\log n} \right) \leq \exp \left( -\Theta \left( \sqrt{\log n} \right) \right) = \exp \left( -\alpha \log n \right) \]

where \( \alpha \) is a function of \( c \).

On the other hand,

\[ \# \left\{ Q^{(i)} \right\} = n^2 \Delta^{5|Q|} \leq n^2 \Delta^{5\sqrt{\log n}} \leq 2^{7\log n} \]

By union bound,

\[ P \left( \exists Q, Q : c \sqrt{\log n} \right) \leq \exp \left( -\alpha \log n \right) 2^{7\log n} \]

If we take \( \alpha \) large, say 100, at least one node of \( P \) is knocked out w.h.p. In the other words, connected components of \( H \) have (mark) diameter at most \( 5\sqrt{\log n} \) w.h.p.