Supporting hyperplane

min $3x + y$ s.t.

a → $x + y \geq 2$

b → $x \geq 0$

c → $y \geq 0$

max $2a$ s.t.

$a + b = 3$

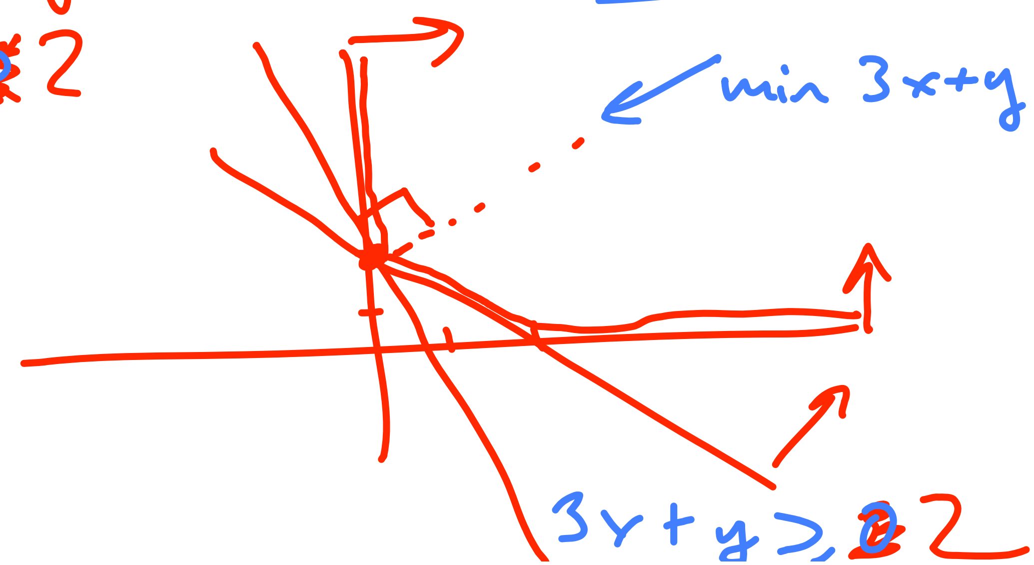
$a + c = 1$

$a, b, c \geq 0$

$x = 0 \quad y = 2$

$a = 1 \quad b = 2 \quad c = 0$

$x + y + 2x \geq \cancel{0}\ 2$



Dual of dual

primal

min $3x + y$ s.t.
$x + y \geq 2$
$x \geq 0$
$y \geq 0$

dual

max $2a$ s.t.
$a + b = 3$
$a + c = 1$
$a, b, c \geq 0$

$x(a+b-3) + y(a+c-1)$

$+ ua + vb + wc \leq 0$

$u, v, w \leq 0$

dual obj $2a =$

$a(x+y+u)$

$+ b(x+v) + c(y+w) \leq \underline{3x+y}$

$x+y+u=2 \Rightarrow \underline{x+y \geq 2}$

$x+v=0 \Rightarrow \underline{x \geq 0}$

$y+w=0 \Rightarrow \underline{y \geq 0}$

The graduate student nutrition problem

min $3x + y$ s.t.	max $2a$ s.t.
$x + y \geq 2 - \epsilon$	$a + b = 3$
$x \geq 0 + \eta$	$a + c = 1$
$y \geq 0$	$a, b, c \geq 0$

val $\downarrow \epsilon$ $\quad x = 0 \quad y = 2 - \epsilon$ $\qquad \underline{a = 1} \quad \underline{b = 2} \quad c = 0$

val $\uparrow 2\eta$ $\quad x = \eta \quad y = 2 - \eta$

Sensitivity analysis

primal

$$\min\ 3x + y \text{ s.t.}$$
$$x + y \geq 2$$
$$x, y \geq 0$$

The Lagrangian

- $L(a,b,c,x,y) =$

constraints

objective $[3x + y] - [a\,(x + y - 2) + bx + cy]$

- $\min_{x,y} \max_{a,b,c\geq 0} L(a,b,c,x,y)$

dual

$$\max\ 2a \text{ st}$$
$$a + b = 3$$
$$a + c = 1$$
$$a, b, c \geq 0$$

$$\frac{d}{dx} L = 3 - a - b = 0$$

$$\frac{d}{dy} L = 1 - a - c = 0$$

$$L = \cancel{\ldots}$$
$$\cancel{x(3 - a - b)}$$
$$\cancel{+\, y(1 - a - c)}$$
$$+\, 2a$$

Lagrangian cont'd

$$\min\ 3x + y \ \text{ s.t.}$$
$$x + y \geq 2$$
$$x, y \geq 0$$

- $L(a,b,c,x,y) =$

  $$[3x + y] - [\cancel{a\,(x + y - 2)} + \cancel{bx} + \cancel{cy}]$$

- $\min_{x,y} \max_{a,b,c\geq 0} L(a,b,c,x,y)$

$$\max\ 2a \text{ st}$$
$$a + b = 3$$
$$a + c = 1$$
$$a, b, c \geq 0$$

$a = 0 \quad d/da\, L \leq 0 \Rightarrow 2 \leq x + y$

$a > 0 \quad \frac{d}{da} L = -x - y + 2 = 0$

$\Big\}\ a(x+y-2) = 0$

$b = 0 \quad d/db\, L \leq 0 \Rightarrow x \geq 0$

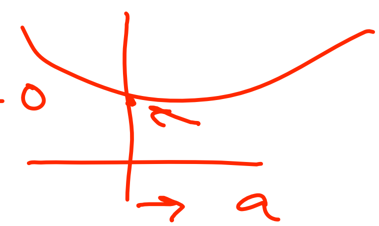
$b > 0 \quad \frac{d}{db} L = -x = 0$

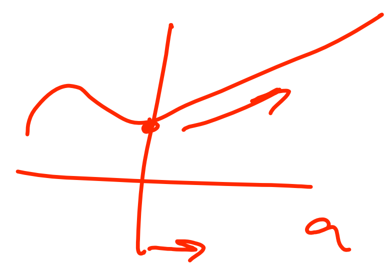
$\Big\}\ bx = 0$

$c = 0$

$c > 0$

$\Big\}\ y \geq 0$



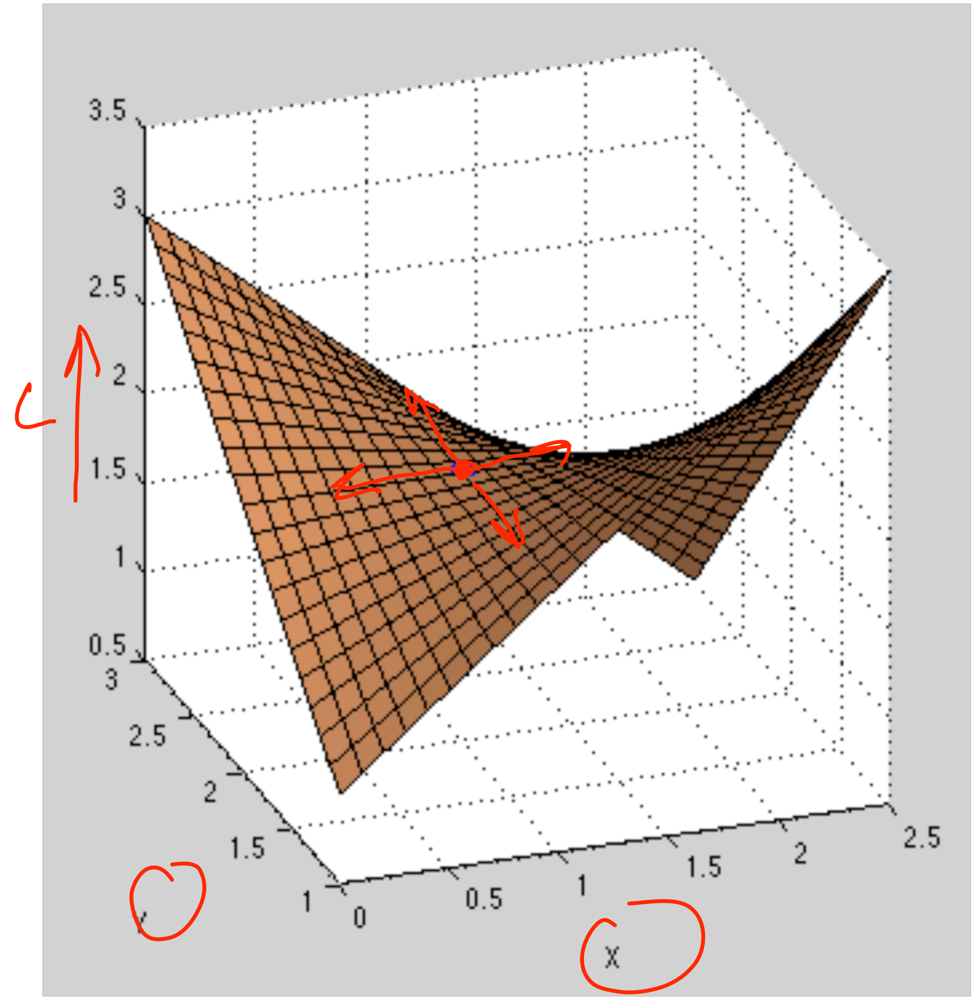


Saddle-point picture

- min y s.t. y ≥ 2

$L = y - x(y-2)$

$(x \geq 0)$



+ vs – in Lagrangian

min y s.t. 2 ≤ y ≤ 4

$$\min_{y} \max_{a,b \geq 0} \; y - [a(y-2) + b(4-y)]$$

$y =$	1	3	5
$a =$	∞	0	0
$b =$	0	0	∞
$L =$	∞	3	∞

max y s.t. 2 ≤ y ≤ 4

$$\max_{y} \min_{a,b \geq 0} \; y + [a(y-2) + b(4-y)]$$

$y =$	1	3	5
$a =$	∞	0	0
$b =$	0	0	∞
$L =$	$-\infty$	3	$-\infty$

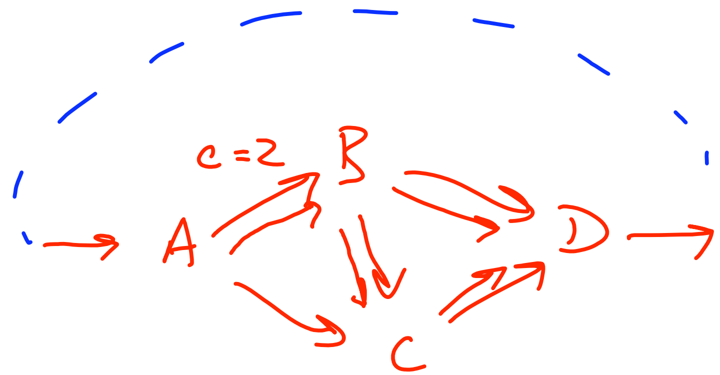
Duality summary

Primal	Dual
min problem	max
max problem	min
constraint	variable (multiplier)
≤ constraint	-ve var
≥ constraint	+ve var
= constraint	free variable
variable	constraint

Duality summary

Primal	Dual
tight constraint	possibly nonzero var
slack constraint	0 dual var
zero/nonzero variable	slack/tight constr
infeasible problem	unbounded
unbounded problem	infeasible
finite optimal value	finite opt

Example: max flow



- Given a directed graph
  - edges $(i,j) \in E$
  - flows f_{ij}, capacities c_{ij}
  - source s, terminal t ($c_{ts} = \infty$)
- max f_{ts} s.t.
  - positive flow $f_{ij} \geq 0 \quad \forall i,j \in E$
  - capacity $f_{ij} \leq c_{ij} \quad \forall i,j \in E \quad i,j \neq t,s$
  - flow conservation $\sum_i f_{ij} = \sum_i f_{ji} \quad \forall j$

Dual of max flow

$\max f_{ts}$

$f_{ij} \geq 0 \nearrow a_{ij}$

$f_{ij} \leq c_{ij} \nearrow b_{ij}$

$\left(\sum_i f_{ij}\right) = \left(\sum_i f_{ji}\right) \nearrow x_j$

$$\max_f \; \min_{\substack{x \\ a,b \geq 0}} L = f_{ts} + \Big[\sum_j x_j \Big(\sum_i f_{ji} - \sum_i f_{ij} \Big) + \sum_{ij} a_{ij} f_{ij} + \sum_{\substack{ij \\ \neq ts}} b_{ij} (c_{ij} - f_{ij}) \Big]$$

$$\left.\begin{aligned} \frac{d}{d f_{ts}} L &= 1 + a_{ts} - x_s + x_t = 0 \\ \frac{d}{d f_{ij}} L &= a_{ij} - b_{ij} - x_j + x_i = 0 \end{aligned}\right\} \quad L = \sum_{ij \neq ts} b_{ij} c_{ij}$$

$i,j \neq t,s$

$$\left.\begin{aligned} &\min_{\substack{x \\ a,b \geq 0}} \sum_{ij \neq ts} b_{ij} c_{ij} \quad \text{s.t.} \\ &1 \leq x_s - x_t \\ &x_i \leq x_j + b_{ij} \end{aligned}\right] \text{dual}$$

Interpreting dual

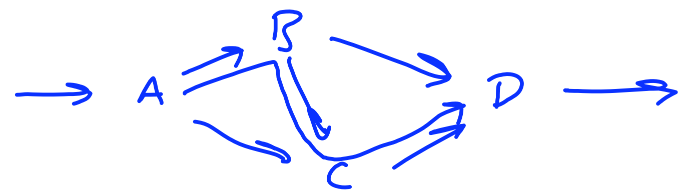
$$\min_{x} \sum_{ij \neq ts} \underline{b_{ij} c_{ij}}$$

$\cancel{x}_s \geq 0$

$$x_t + 1 \le x_s \Rightarrow x_s \ge 1$$

$$x_i \le x_j + b_{ij}$$

$$x_t = 0$$



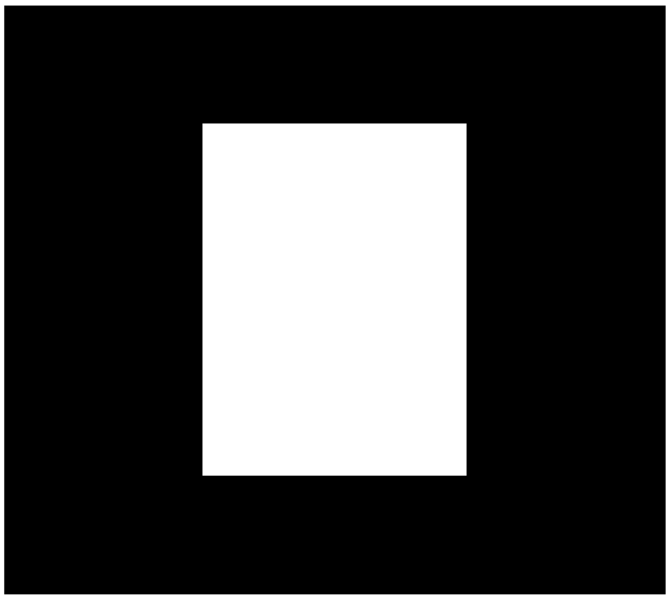
$$x_A \overset{\le}{\cancel{\ge}} x_B + b_{AB}$$

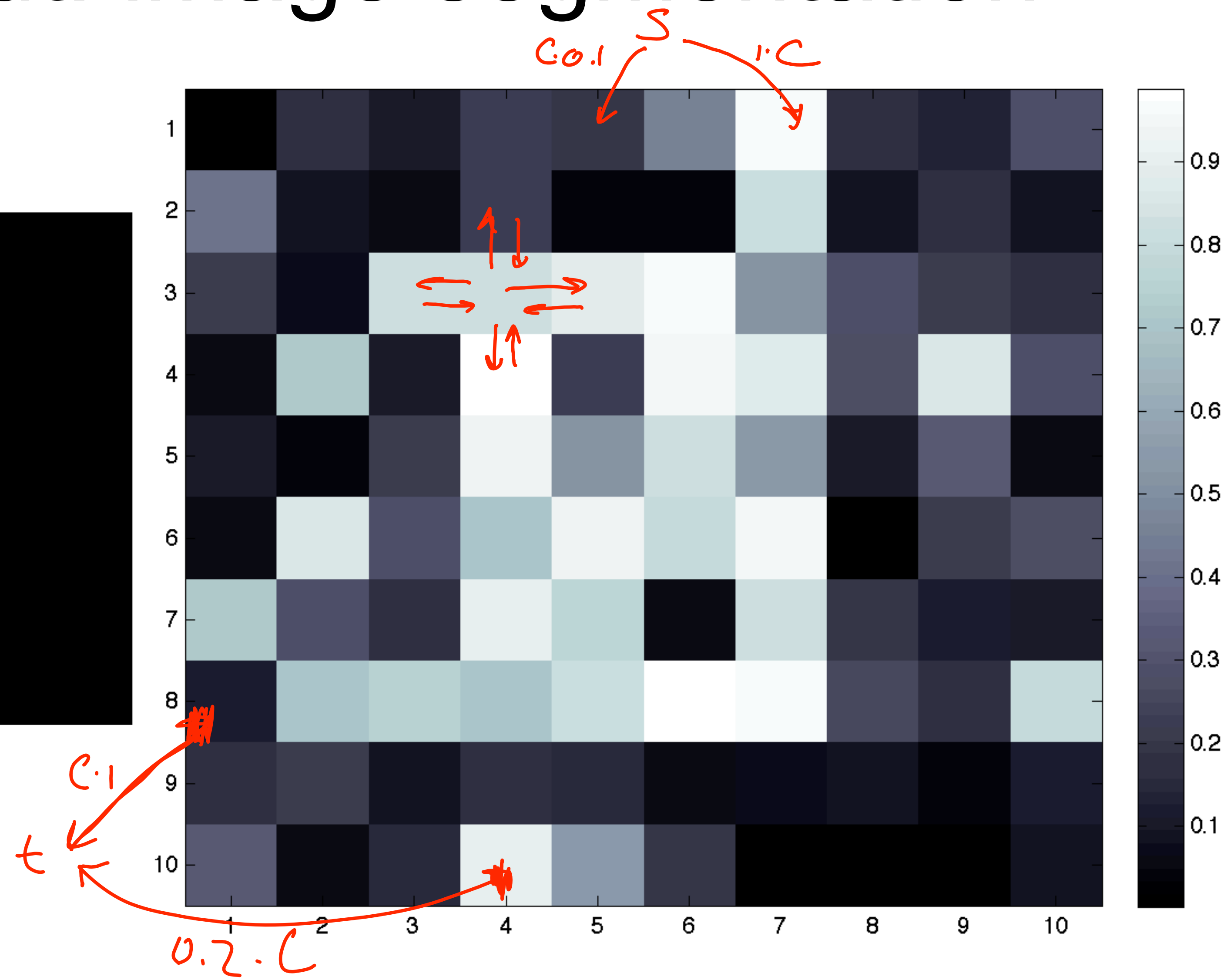
$$x_B \overset{\le}{\cancel{\ge}} x_C + b_{BC}$$

$$x_C \overset{\le}{\cancel{\ge}} x_D + b_{CD}$$

$$1 \le x_A \le \underbrace{\cancel{x_D}}_{=0} + b_{AB} + b_{BC} + b_{CD}$$

min cut: image segmentation





Solution