Complementary slackness

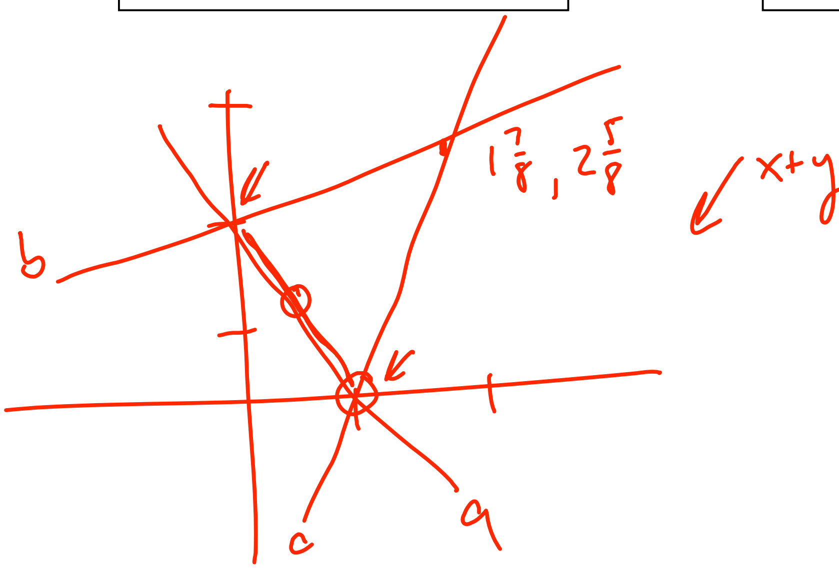
2x+y

$$\begin{array}{lr} & \min\ x + y \\ a & 2x + y \geq 2 \\ b & x - 3y + 6 \geq 0 \\ c & -3x + y + 3 \geq 0 \end{array}$$

x=1
y=0

$$\begin{array}{r} \max\ 2a - 6b - 3c \\ \overset{=2}{2a + b - 3c = 1} \\ a - 3b + c = 1 \\ a, b, \underset{=1}{c} \geq 0 \end{array}$$

a = 4/5
b = 0
c = 1/5



2a = 2
a = 1

Complementary slackness

$$\min \quad x + y$$

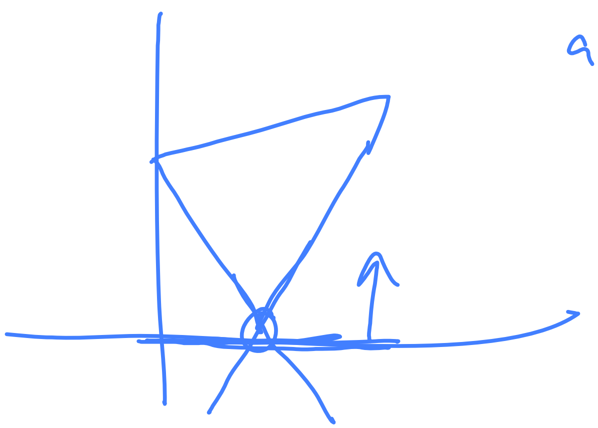
a	$2x + y \ge 2$
b	$x - 3y + 6 \ge 0$
c	$-3x + y + 3 \ge 0$
d	$y \ge 0$

$$\max \quad 2a - 6b - 3c$$

$$2a + b - 3c = 1$$

$$a - 3b + c + d = 1$$

$$a, b, c, d \ge 0$$



a, c, d tight

dual opt

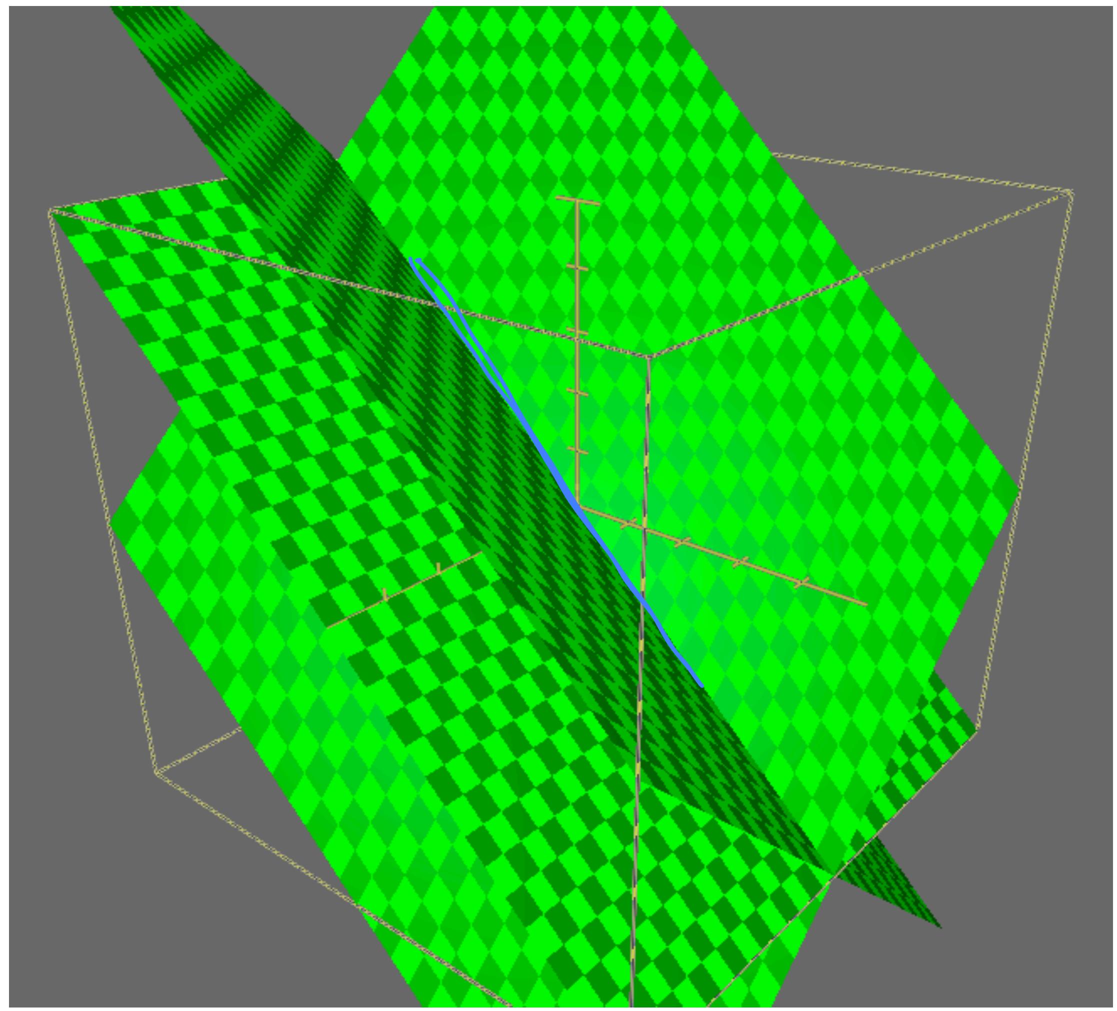
$4/5, \; 0, \; \frac{1}{5}, \; 0$

a b c d

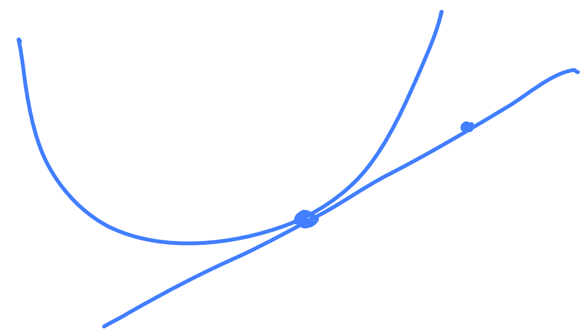
Complementary slackness in 3D

min $-y + z$ s.t.

$$-x - y + z \geq 0$$
$$x - y + z \geq 0$$
$$-y + z \geq 0$$



What about QP duality?



- min $x^2 + y^2$ s.t.

  $\mathcal{C}\left\{\begin{array}{l} x + 2y \geq 2 \\ x, y \geq 0 \end{array}\right.$

- How can we lower-bound OPT?

$$x^2 + y^2 \geq 2 + 2(x-1) + 2(y-1) = \underset{\text{s.t. } \mathcal{C}}{\overset{\min}{2x + 2y - 2}}$$

$a(x+2y-2) + bx + cy \geq 0$ if $a,b,c \geq 0$

$x(a+b) + y(2a+c) - 2 \geq 2a - 2$ if $a,b,c \geq 0$

$2x + 2y - 2 = x(a+b) + y(2a+c) - 2$

if $a+b=2$, $2a+c=2$

so, $x^2+y^2 \geq 2a - 2$ if

$a+b = 2$

$2a+c = 2$

$a,b,c \geq 0$

Works at other points too

- min $x^2 + y^2$ s.t.

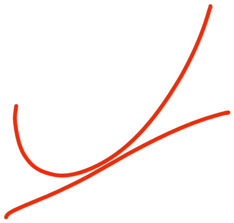
  $x + 2y \geq 2$

  $x, y \geq 0$
- Try Taylor @ $(x, y) = (v, w)$

$$\min \ x^2+y^2 \quad \longrightarrow \quad x^2+y^2 \geq v^2+w^2 + 2v(x-v) + 2w(y-w)$$

$$\text{s.t. } x+2y \geq 2 \qquad = 2vx + 2wy - v^2 - w^2$$

$$x, y \geq 0$$



$$a(x+2y-2) + bx + cy \geq 0$$

$$x(a+b) + y(2a+c) - 2a \geq 0$$

$$\rightarrow \quad x(a+b) + y(2a+c) - v^2 - w^2 \geq 2a - v^2 - w^2$$

DUAL

$$\max \ 2a - v^2 - w^2$$

$$\text{s.t. } a+b = 2v$$

$$2a+c = 2w$$

$$a, b, c \geq 0$$

$$\text{if } \left\{ \begin{array}{l} a+b = 2v \\ 2a+c = 2w \end{array} \right\} \text{ then}$$

$$x^2+y^2 \geq 2vx + 2wy - v^2 - w^2 = x(a+b) + y(2a+c) - v^2 - w^2$$

$$\geq 2a - v^2 - w^2$$

SVM duality

- Recall: min $\underset{\mathbb{R}^n}{\|w\|^2}/2$ s.t. $\underset{\pm 1}{y_i}(\underset{\mathbb{R}^n}{x_i}\cdot w - \underset{\mathbb{R}}{b}) \geq 1$
- Taylor bound objective: $\|w\|^2/2 \geq \frac{\|v\|^2}{2} + (w-v)\cdot v$
  $= w\cdot \underline{v} - \|v\|^2/2$
- Generic constraint:

$$\sum_i \alpha_i (y_i(x_i\cdot w - b) - 1) \geq 0$$

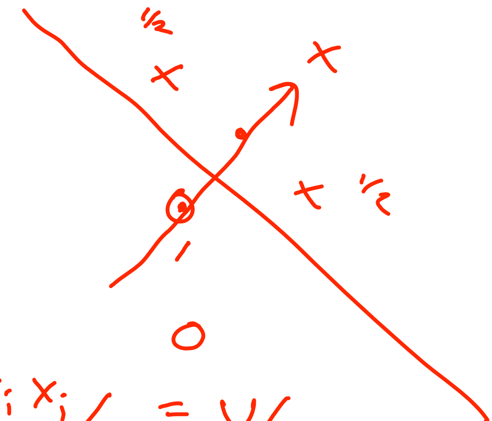
$$-\frac{\|v\|^2}{2} + w\cdot \underline{\left(\sum_i \alpha_i y_i x_i\right)} - b\underline{\sum_i \alpha_i y_i} \geq \underline{\sum_i \alpha_i - \|v\|^2/2}$$

- To get bound, need:

$$v = \sum_i \alpha_i y_i x_i \qquad \max \sum_i \alpha_i - \|v\|^2/2$$

$$0 = \sum_i \alpha_i y_i \qquad \alpha_i \geq 0$$

SVM dual



- $\max_{\alpha,v} \Sigma_i \alpha_i - ||v||^2/2$ s.t.

  $\underline{\Sigma_i \alpha_i y_i = 0}$ $\quad Z = \sum_{i|y_i=1} \alpha_i = \sum_{i|y_i=-1} \alpha_i$

  $\underline{\Sigma_i \alpha_i y_i x_{ij} = v_j}$ $\quad \forall j$ $\quad \underbrace{\sum_{i|y_i=1} \frac{\alpha_i x_i}{Z}} - \underbrace{\sum_{i|y_i=-1} \alpha_i x_i / Z} = v/Z$

  $\alpha_i \geq 0$ $\quad \forall i$

convex combo of +ve x_i — convex combo of −ve x_i