Submodular functions cont.

Optimization - 10725

Carlos Guestrin
Carnegie Mellon University

April 30th, 2008

Submodularity

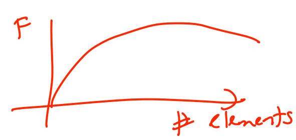
supermodular $F(A \cup \{x\}) - F(A) \leq F(B \cup \{x\}) - F(B)$, $A \subseteq B$

modular $F(A \cup \{x\}) - F(A) = F(B \cup \{x\}) - F(B)$ $\iff F(A) = \sum_{A_i \in A} F(A_i)$

- Formalizes notion of diminishing returns

$\forall A, B$ $\quad F(A \cup \{x\}) - F(A) \geq F(B \cup \{x\}) - F(B)$

$A \subseteq B$



- Equivalent definition:

$\forall A, B \quad F(A) + F(B) \geq F(A \cup B) + F(A \cap B)$

What do we get from submodularity

- Submodularity is a general property of set functions
- Submodular function can be minimized in polynomial time!

$$\min_{A \subseteq V} F(A)$$

F is submodular — polytime

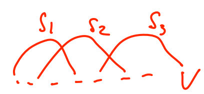
- But our problem is

$$\max_{A,\ |A| \le k} F(A)$$

F is submodular

Another example…

- Maximum cover
  - Ground elements $v \in V$
  - Set of sets $S_i \subseteq V$
  - Pick k sets, maximize number of covered elements



$$F(A) = \sum_v \mathbb{I}\left(v \text{ is } \bigcup_{i \in A} S_i\right)$$

A is a set of sets

$A = \{S_1, S_2, \dots S_k\}$

$$F(A \cup \{x\}) - F(A) \geq F(B \cup \{x\}) - F(B)$$

$A \subseteq B$

consider element v

- covered A
- covered by x, but not B or A
- covered B, but not A

Maximizing submodular functions – cardinality constraint

- Given
  - Submodular function $F(A)$
  - Normalized $F(\emptyset) = 0$
  - Non-decreasing $F(A) \leq F(B)$ $A \subseteq B$

  e.g.) max cover

- Greedy algorithm guarantees

$$F(A_{Greedy}) \geq (1 - \frac{1}{e}) F(A_{OPT})$$

$(1 - \frac{1}{e})$: 63%; $F(A_{OPT}) = \max_{A, |A| \leq k} F(A)$

- Can you get better algorithm?

NO, unless P=NP, if you could solve max-cover better than $(1-1/e)$ then P=NP

©2008 Carlos Guestrin

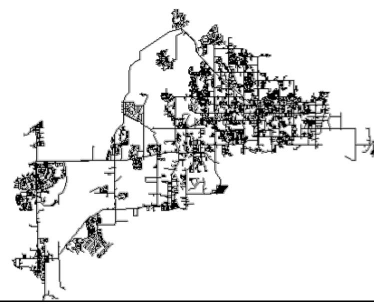
Online bounds

- Submodularity provides bounds on the quality of the solution A obtained by any algorithm
  - For normalized, non-decreasing functions
- Advantage of adding elements to A:
- Bound on the quality of any set A:
- Tighter bound:

©2008 Carlos Guestrin

Battle of the Water Sensor Networks Competition

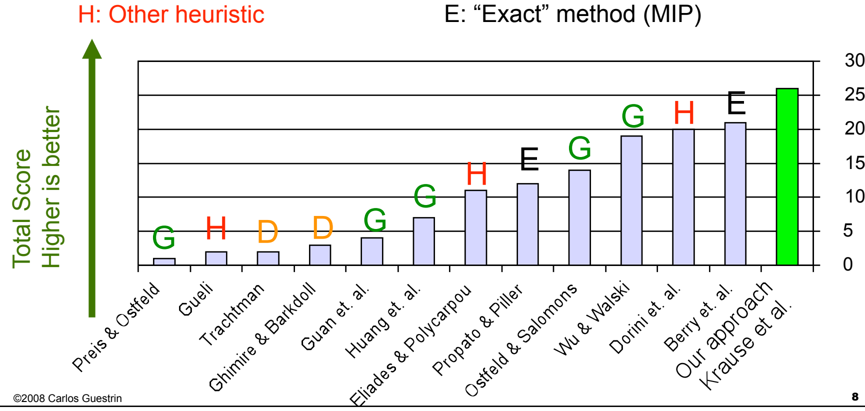
- Real metropolitan area network (12,527 nodes)
- Water flow simulator provided by EPA
- 3.6 million contamination events
- Multiple objectives: Detection time, affected population, …
- Place sensors that detect well "on average"



BWSN Competition results

- 13 participants
- Performance measured in 30 different criteria

G: Genetic algorithm
D: Domain knowledge
H: Other heuristic
E: "Exact" method (MIP)

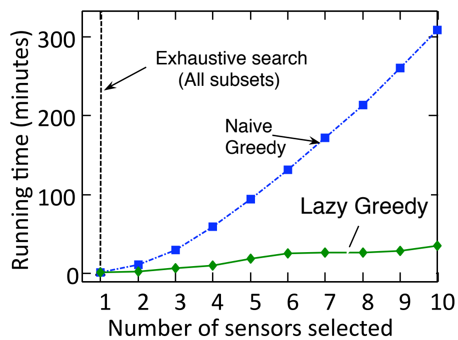


What was the trick?

Simulated all 3.6M contaminations on 2 weeks / 40 processors

152 GB data on disk, 16 GB in main memory (compressed)

➔ Very accurate sensing quality ☺ Very slow evaluation of F(A) ☹



30 hours/20 sensors

6 weeks for all

30 settings ☹

ubmodularity

to the rescue:

Using "lazy evaluations":

1 hour/20 sensors

Done after 2 days! ☺

 9

Lazy evaluations

- Naïve implementation of greedy:
- Advantage of an element never increases:
  - Advantage:
  - What if you already picked a larger set:
    - Set after picking i elements: A_i
- Lazy evaluations:
  - Keep a priority queue over elements:
    - Initialize with advantage of each element
  - Pick element on top, recompute priority
  - If element remains on top

 10

Other maximization settings

- Non-monotone submodular functions
- Non-unit costs
- Complex constraints
  - Paths
  - Spanning trees
- Worst-case optimization

Announcements

- University Course Assessments
  - Please, please, please, please, please, please, please, please, please, please, please, please, please, please, please, please…
- Project:
  - Poster session: Tomorrow 3-6pm, NSH Atrium
    - please arrive a 15mins early to set up
    - Don't wait until the last minute to print
  - Paper: May 5th by 3pm
    - electronic submission by email to instructors list
    - maximum of 8 pages, NIPS format
    - no late days allowed
- Final:
  - Out: Monday, May 5
  - Due: Friday, May 9

Submodularity and concavity

- Consider set function F(A) that only depends |A|:
- Recall defn of submodular functions:
- In fact, F(|A|) submodular if and only if

Submodular polyhedron

- |V|=n, for x in R^n
  - Define X(A) =
- Submodular polyhedron
- For positive costs c, suppose we maximize:

Maximizing over submodular polyhedron

- Want:

- Complex polyhedron, but very simple solution
  - Order nodes in increasing order of cost:
  - OPT x:
    -
    -
    - Prove optimality using duality

Extension of a set function

- For any set function F, define extension of F by:

- Easy to compute for submodular functions

- Amazing Theorem:

What can we do with convexity of extension?

- Suppose c_A is a 01-vector for set A:
  - c(i) =
- Formulate maximization:
- At optimum:
  - By telescopic sum using OPT x

Minimizing submodular functions

- We know that
- Integer program:
- Convex relaxation
  - At optimum,
  - Can be solved using
- Thus, submodular function minimization:

Minimizing symmetric submodular fns

- Minimizing general submodular fns, not practical, because of ellipsoid algorithm
- Symmetric submodular fns:
- If submodular function symmetric:
- Want non-trivial minimum:
- Queryanne's Algorithm for minimizing symmetric submodular fns:
  - Very simple to implement
  - Only $O(n^3)$

Application of minimizing symmetric submodular fns

- Given set V of random variables
  - Split into two sets that are as independent as possible
- Submodular function:
- Can be optimized using Queyranne's algorithm
  - Useful, e.g., structure learning in graphical models

Submodular fns overview

- Minimized in polytime
- Approximate maximization
  - Under many different constrained settings
- Many many applications in AI/ML
  - Structure learning in graphical models
  - Clustering
  - Active learning
  - Sensor placement
  - Viral marketing
  - What blogs should we read to stay in touch with important stories
  - …
- Not explored enough, plenty of opportunities!!