

Want to solve integer program E.g., vars in {0,1} Solve convex relaxation E.g., vars in [0,1] If minimizing, relaxed objective lower: Want integer solution: Somehow round relaxed solution: Can affect feasibility Can affect costs Today: some ideas & strategies for rounding

☐ See optional books for many more options & details

Integral basic feasible solutions



- If all optimal basic feasible solutions are integral, we are done!
 - □ LP relaxation is optimal!!!
- It is sufficient if all basic feasible solutions are integral
 - □ When does this happen?
 - □ A sufficient (but not necessary) condition:

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Integral matrix →Integral inverse?



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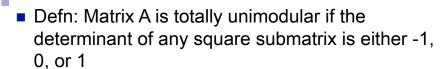
One sufficient (but not necessary) condition: Totally Unimodular matrix

Structure of inverse of matrix:

- Inverse integral if
 - □ Determinant:
 - □ Cofactors:

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Relaxations with Totally Unimodular Matrices



- Thm: If an LP has a totally unimodular constraint matrix A, and the vector b is integral, then all basic feasible solutions are integral
 - □ Thus

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How often do you see totally unimodularity?

- Often
 - □ Bipartite matching
 - □ Cuts
 - □ Maximum margin Markov networks
- Not often

One thing we can agree: it's usually not easy to spot...

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Sufficient conditions for total unimodularity



- Matrix A is totally unimodular if
 - □ All entries are -1, 0, or 1
 - □ Each column contains at most two nonzero elements
 - Rows of A can be partitioned into two sets A₁ and A₂ such that two nonzero entries in a column are
 - in the same set of rows if they have different signs
 - in different sets of rows if they have the same sign
- Maximum bipartite matching:
 - □ Two sets of nodes
 - Edges from nodes i in A to j in B have weight w_{ij}

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Relaxations and rounding



What do we do if we don't get integral solutions?

- E.g., set cover problem
 - □ Ground elements
 - □ Set of Sets
 - □ Cost for sets
 - □ Find cheapest collection of subsets that covers all elements
- Integer program and relaxation:
- How can we obtain a good integer (rounded) solution?
 - $\hfill\Box$ If we set all nonzero x_S to one, then

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Consider a special case...



- Suppose each element in at most k sets
- From inequality constraint:
- Rounding strategy:
- Feasibility:
- Cost of rounded solution:

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Very simple example of randomized rounding

- Solve set cover relaxation:
- Randomly pick a collection of subsets G
 For each S, add it to G with (independent) probability x_s
- What's the expected cost of G?
 - □ I_s indicator of whether set S is in G

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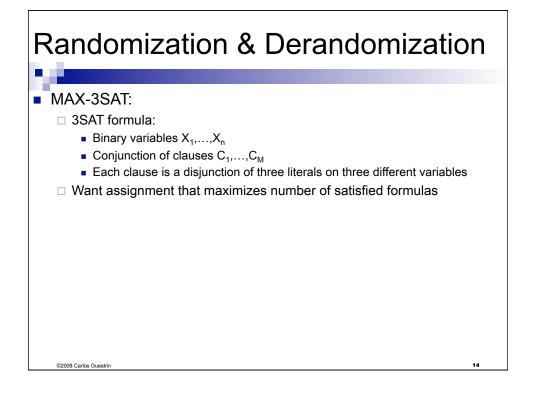
How many elements do we cover?



- Expected cost of G can be lower than OPT_{IP}
 - □ Must cover fewer elements
- I_v is indicator of whether element v covered by G
- Expected number of elements covered:

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How big can cost get? Expected cost is lower than OPT_{IP} But how big can actual cost get? (a simple bound here, more interesting bounds using more elaborate techniques) Markov Inequality: Let Y be a non-negative random variable Then In our example:



Randomized algorithm for MAX-3SAT



- Pick assignment for each X_i independently, at random with prob. 0.5
- Expected number of satisfied clauses:

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Aside: Probabilistic Method



- Expected number of satisfied clauses:
- Probabilistic method: for any random var. Y, there exists assignment y such that P(y)>0, y≥E[Y]
 - □ Almost obvious fact
 - □ Amazing consequences
- For example, in the context of MAX-3SAT:

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Derandomization



- There exists assignment X that achieves
- In expectation, we get 7/8.M, but can we get it with prob. 1? Without randomization?
- Derandomization: From a randomized algorithm, obtain a deterministic algorithm with same guarantees
 - □ Today: method of conditional expectations

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Method of conditional expectations



- Conditional expectation:
- Expectation of the conditional expectation:

- Consider MAX-3SAT:
 - Expectation:
 - □ Expectation of conditional expectation:

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Computing conditional expectation



- Conditioning on X₁=1:
- General case: $X_1 = v_1, ..., X_i = v_i$
 - $\hfill \square$ Sum over clauses, $\hfill I_i$ is indicator clause j is satisfied

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Derandomized algorithm for MAX-3SAT



- For i=1,...,n
 - □ Try X_i=1
 - Compute
 - □ Try X_i=0
 - Compute
 - $\ \square$ Set v_i to best assignment to X_i
- Deterministic algorithm guaranteed to achieve at least 7/8.M

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