From relaxations to integral solutions

Optimization - 10725
Carlos Guestrin
Carnegie Mellon University

April 21st, 2008

©2008 Carlos Guestrin

Today…

- Want to solve integer program
  - E.g., vars in {0,1}
- Solve convex relaxation
  - E.g., vars in [0,1]
- If minimizing, relaxed objective lower:
- Want integer solution:
  - Somehow round relaxed solution:
    - Can affect feasibility
    - Can affect costs
- Today: some ideas & strategies for rounding
  - See optional books for many more options & details

©2008 Carlos Guestrin

Integral basic feasible solutions

- LP:
- If all optimal basic feasible solutions are integral, we are done!
  - LP relaxation is optimal!!!
- It is sufficient if all basic feasible solutions are integral
  - When does this happen?
  - A sufficient (but not necessary) condition:

©2008 Carlos Guestrin

Integral matrix → Integral inverse?

©2008 Carlos Guestrin

One sufficient (but not necessary) condition: Totally Unimodular matrix

- Structure of inverse of matrix:
- Inverse integral if
  - Determinant:
  - Cofactors:

©2008 Carlos Guestrin

Relaxations with Totally Unimodular Matrices

- Defn: Matrix A is totally unimodular if the determinant of any square submatrix is either -1, 0, or 1
- Thm: If an LP has a totally unimodular constraint matrix A, and the vector b is integral, then all basic feasible solutions are integral
  - Thus

©2008 Carlos Guestrin

How often do you see totally unimodularity?

- Often
  - Bipartite matching
  - Cuts
  - Maximum margin Markov networks
- Not often
  -
- One thing we can agree: it's usually not easy to spot…

©2008 Carlos Guestrin

Sufficient conditions for total unimodularity

- Matrix A is totally unimodular if
  - All entries are -1, 0, or 1
  - Each column contains at most two nonzero elements
  - Rows of A can be partitioned into two sets $A_1$and A_2 such that two nonzero entries in a column are
    - in the same set of rows if they have different signs
    - in different sets of rows if they have the same sign
- Maximum bipartite matching:
  - Two sets of nodes
    - Edges from nodes i in A to j in B have weight w_{ij}

©2008 Carlos Guestrin

Relaxations and rounding

- What do we do if we don't get integral solutions?
  -
- E.g., set cover problem
  - Ground elements
  - Set of Sets
  - Cost for sets
  - Find cheapest collection of subsets that covers all elements
- Integer program and relaxation:
- How can we obtain a good integer (rounded) solution?
  - If we set all nonzero x_S to one, then
  -

©2008 Carlos Guestrin

Consider a special case…

- Suppose each element in at most k sets
- From inequality constraint:
- Rounding strategy:
- Feasibility:
- Cost of rounded solution:

©2008 Carlos Guestrin

Very simple example of randomized rounding

- Solve set cover relaxation:
- Randomly pick a collection of subsets G
  - For each S, add it to G with (independent) probability x_s
- What's the expected cost of G?
  - I_s indicator of whether set S is in G

©2008 Carlos Guestrin

How many elements do we cover?

- Expected cost of G can be lower than OPT_{IP}
  - Must cover fewer elements
- I_v is indicator of whether element v covered by G
- Expected number of elements covered:

©2008 Carlos Guestrin

How big can cost get?

- Expected cost is lower than OPT_{IP}
  - But how big can actual cost get?
  - (a simple bound here, more interesting bounds using more elaborate techniques)
- Markov Inequality: Let Y be a non-negative random variable
  - Then
- In our example:

©2008 Carlos Guestrin

Randomization & Derandomization

- MAX-3SAT:
  - 3SAT formula:
    - Binary variables $X_1,\ldots,X_n$
    - Conjunction of clauses $C_1,\ldots,C_M$
    - Each clause is a disjunction of three literals on three different variables
  - Want assignment that maximizes number of satisfied formulas

©2008 Carlos Guestrin

Randomized algorithm for MAX-3SAT

- Pick assignment for each X_i independently, at random with prob. 0.5
- Expected number of satisfied clauses:

©2008 Carlos Guestrin

Aside: Probabilistic Method

- Expected number of satisfied clauses:
- Probabilistic method: for any random var. Y, there exists assignment y such that P(y)>0, y≥E[Y]
  - Almost obvious fact
  - Amazing consequences
- For example, in the context of MAX-3SAT:

©2008 Carlos Guestrin

Derandomization

- There exists assignment X that achieves
- In expectation, we get 7/8.M, but can we get it with prob. 1? Without randomization?
- Derandomization: From a randomized algorithm, obtain a deterministic algorithm with same guarantees
  - Today: method of conditional expectations

©2008 Carlos Guestrin

Method of conditional expectations

- Conditional expectation:
- Expectation of the conditional expectation:
  -
- Consider MAX-3SAT:
  - Expectation:
  - Expectation of conditional expectation:

©2008 Carlos Guestrin

Computing conditional expectation

- Conditioning on $X_1=1$:

- General case: $X_1=v_1,\ldots, X_i=v_i$
  - Sum over clauses, I_j is indicator clause j is satisfied

©2008 Carlos Guestrin

Derandomized algorithm for MAX-3SAT

- For i=1,…,n
  - Try $X_i=1$
    - Compute
  - Try $X_i=0$
    - Compute
  - Set v_i to best assignment to X_i
- Deterministic algorithm guaranteed to achieve at least 7/8.M

©2008 Carlos Guestrin