From relaxations to integral solutions

Optimization - 10725

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Today…

- Want to solve integer program
  - E.g., vars in {0,1}
- Solve convex relaxation
  - E.g., vars in [0,1]
- If minimizing, relaxed objective lower:

OPT_{LP} OPT_{IP} Rounded solution

"small"

$\min_x f_0(x)$

$f_i(x) \leq 0$

$x_i \in$ integer $\{0,1\}$

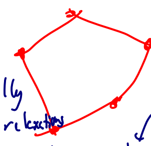
$0 \leq x_i \leq 1$

- Want integer solution:
  - Somehow round relaxed solution:
    - Can affect feasibility
    - Can affect costs
- Today: some ideas & strategies for rounding
  - See optional books for many more options & details

Integral basic feasible solutions

- LP: $\min_x c'x$ $\quad Ax \geq b \quad 0 \leq x \leq 1$ $\Big\}$ always get integer solution?

- If all optimal basic feasible solutions are integral, we are done!
  - LP relaxation is optimal!!!
- It is sufficient if all basic feasible solutions are integral
  - When does this happen?
  - A sufficient (but not necessary) condition: trivial to check and usually true for relaxation



basis $B \subseteq$ rows of A

$A_B x = b_B$ $\Big\}$ $x = \underbrace{A_B^{-1}}_{\text{integral}} \underbrace{b_B}_{\text{integral}}$ ← corners, vertices

integral ← typically, we know A integral e.g. $-1, 0, 1$

Integral matrix (A_B) → Integral inverse (A_B^{-1})?

$$A_B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad A_B^{-1} = \frac{1}{|A_B|} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$|A_B| = 2 \qquad\qquad = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

e.g., $b_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad x = A_B^{-1} b_B = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$ not integral

Key problem in e.g., $|A_B| > 1$

One sufficient (but not necessary) condition: Totally Unimodular matrix

- Structure of inverse of matrix:

$$D^{-1} = \frac{1}{|D|}\begin{pmatrix} c_{11} & \cdots & c_{n1} \\ \vdots & & \vdots \\ c_{1n} & \cdots & c_{nn} \end{pmatrix}$$ where c_{ij} is the ij cofactor

$$c_{ij} = (-1)^{i+j}\begin{vmatrix} D_{1,1} & \cdots & D_{1,j-1} & D_{1,j+1} & \cdots & D_{1,n} \\ \vdots & & & & & \\ D_{i-1,1} & & & & & \\ D_{i+1,1} & & \ddots & & & \\ \vdots & & & \ddots & & \\ D_{n,1} & & & & \ddots & \end{vmatrix}$$

- Inverse integral if
  - Determinant: $|D| \leftarrow \{-1, 0, 1\}$
  - Cofactors: $-1, 0, 1$ determinant of square submatrices

Relaxations with Totally Unimodular Matrices

- Defn: Matrix A is totally unimodular if the determinant of ~~any~~ every square submatrix is either -1, 0, or 1

$$A = \begin{pmatrix} \boxed{} \end{pmatrix}$$

- Thm: If an LP has a totally unimodular constraint matrix A, and the vector b is integral, then all basic feasible solutions are integral
  - Thus LP relaxation provides solution to integer program

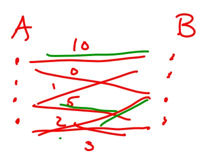
How often do you see totally unimodularity?

- Often
  - Bipartite matching
  - Cuts
  - Maximum margin Markov networks
- Not often
  - otherwise P=NP
- One thing we can agree: it's usually not easy to spot…

Sufficient conditions for total unimodularity

- Matrix A is totally unimodular if
  - All entries are -1, 0, or 1
  - Each column contains at most two nonzero elements

    a variable is in at most two constraints
  - Rows of A can be partitioned into two sets A_1 and A_2 such that two nonzero entries in a column are
    - in the same set of rows if they have different signs
    - in different sets of rows if they have the same sign
- Maximum bipartite matching:
  - Two sets of nodes
    - Edges from nodes i in A to j in B have weight w_{ij}



weight

$$\max_{x} \sum_{ij} c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_j x_{ij} = 1 \quad i \in A$$

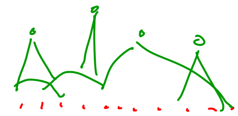
$$\sum_i x_{ij} = 1 \quad j \in B$$

$$x_{ij} \in \{0,1\} \quad \leftarrow \text{relaxation} \quad x_{ij} \geq 0$$

totally unimodular matrix
✓ all entries 0,1
✓ column x_{ij} appear twice
✓ $A_1 = A$ $A_2 = B$, all signs are positive

Relaxations and rounding

- What do we do if we don't get integral solutions?
  - because $P \neq NP$ (probably true)
- E.g., set cover problem
  - Ground elements $v \in V$
  - Set of Sets $S \in \mathcal{S}$ $\quad S \subseteq V$
  - Cost for sets c_S
  - Find cheapest collection of subsets that covers all elements



E

- Integer program and relaxation:

$$\min_x \sum_S c_S x_S$$
$$x_S \in \{0,1\}$$
$$\forall v \quad \sum_{S: v \in S} x_S \geq 1$$

relax

$$\min_x \sum_S c_S x_S$$
$$\sum_{S: v \in S} x_S \geq 1 \quad \forall v$$
$$0 \leq x_S \leq 1$$

- How can we obtain a good integer (rounded) solution?
  - If we set all nonzero x_S to one, then very bad idea... arbitrarily $O(n)$ more expensive solution than OPT_{IP}
  - Smart rounding?