Quadratic programs

Problem statement

- Linear = ≥ ≤ constraints
- Quadratic objective

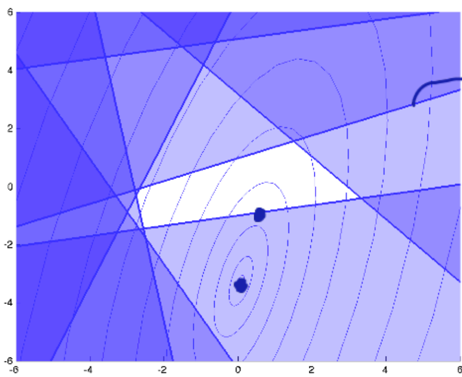
vars x

- minimize $c^T x + \frac{x^T H x}{2}$ subject to

  Ax + b ≥ 0

  Ex + d = 0

For example

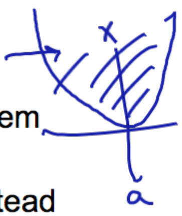


$.6x$
$-1.4y$
≤ 1.4

not a QP

QCQP

$a \geq x^2$



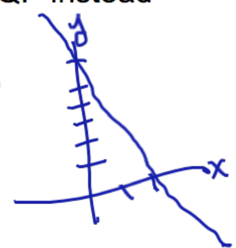
- Quadratic constraints make problem harder -- no longer called a QP!
- Quadratically-constrained QP instead

quadratic constraints

minimize $a^2 + b^2$ subject to

$\quad a \geq x^2,\ \ b \geq y^2$

$\quad 3x + y = 6$

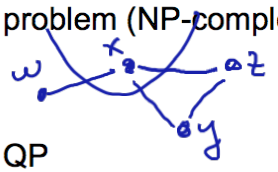


min $x^4 + y^4$

How hard are QPs?

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} = E$$

- Max cut problem (NP-complete)



$V = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$

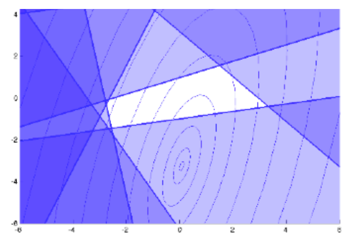
- Max cut QP

$0 \le v \le 1 \quad \max\ v^T E (1-v) \rightarrow$ ex $w(1-x)$

↳ indefinite

- QP is NP-hard (in fact, NP-complete)

Convex QPs

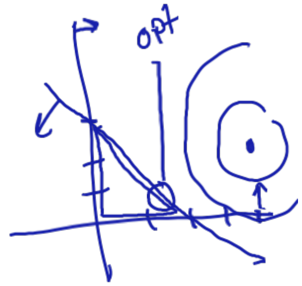


+ve semidef.

- +ve semidefinite objective
  - minimize x'Hx/2 + c'x
- Or -ve semidefinite for maximization
- Convex QP is about as hard as LP (poly-time, but very flexible)

Sketching convex QPs

- min $(x-4)^2 + (y-2)^2$ s.t.
- $3x + 2y \leq 6$
- $x \geq 0$, $y \geq 0$



Sketching ellipses

+ve semidef

- min $x'Hx/2$ - $c'x$ s.t. …
- If H = identity, circles centered @ c
- For positive-definite $H = A'A$:
- min $\underbrace{x'A'}_{y'}\underbrace{Ax}_{y}/2$ - $c'x$ $\qquad$ $y = Ax$ $\qquad$ $x = A^{-1}y$

min $y'y/2 - c'A^{-1}y$

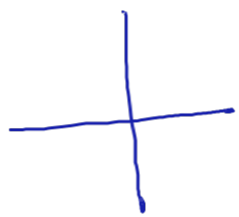
- In y-space, circles around $A^{-1}c$
- In x-space: images of circles under A

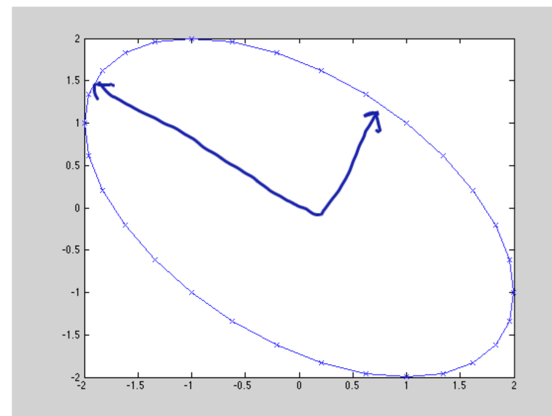
Sketching example

- H = $\begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}$ = L L'
- L = $\begin{pmatrix} 2 & 0 \\ -1 & \sqrt{3} \end{pmatrix}$

$x = L^{-1} y$

y

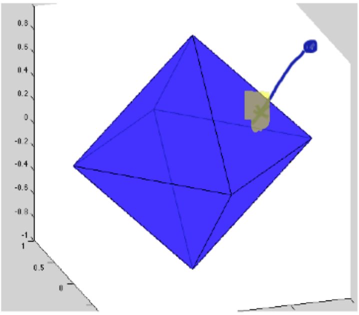




Quadratic program examples

Euclidean projection

- Find point closest to (3, 3, 3) in octahedron



$$\min_{x,y,z} (x-3)^2 + (y-3)^2 + (z-3)^2$$

$$x + y + z \le 1$$
$$x + y - z \le 1$$
$$\vdots$$

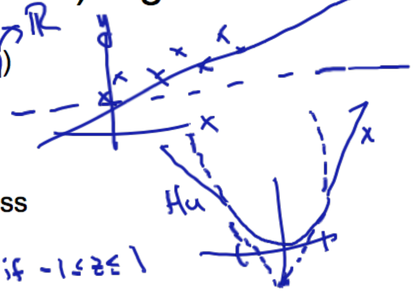
8 total constrs

Robust (Huber) regression

$\mathbb{R}^n \quad \mathbb{R}$

- Given points (x_i, y_i)
- L_2 regression: $\mathbb{R}^n$
  min$_w$ Σ_i $(y_i - x_i'w)^2$
- Problem:
- Solution: Huber loss
  min$_w$ Σ_i Hu$(y_i - x_i'w)$

Hu(z) = $\begin{cases} z^2 & \text{if } -1 \le z \le 1 \\ 2z-1 & \text{if } z \ge 1 \\ -2z-1 & \text{if } z \le -1 \end{cases}$



Huber loss as QP

- Hu(z): $\min_{a,b} (z-a+b)^2 + 2a + 2b = W(a,b)$
  $a, b \ge 0$

try $a=0$ $\quad W(0,b) = (z+b)^2 + 2b$

$\frac{d}{db} W = 2(z+b) + 2 \overset{?}{=} 0 \Rightarrow b = -z-1$ $\quad z \le -1$

$\Rightarrow W(0,b) = (-1)^2 - 2z - 2 = -2z-1$

try $b=0$ $\quad W(a,0) = (z-a)^2 + 2a \quad \frac{d}{da} W = 2(z-a)(-1) + 2 \overset{?}{=} 0$

$W = 2z-1$

$z \ge 1$

try $a=b=0$ $\quad W(0,0) = z^2$ $\quad$ o/w

$$\min \sum_i Hu(y_i - w \cdot x_i)$$

$$\hookrightarrow \min_{w,\, a_i,\, b_i} \sum_i (y_i - w \cdot x_i - a_i + b_i)^2 + 2a_i + 2b_i$$

$a_i \geq 0 \quad \forall i$

$b_i \geq 0 \quad \forall i$

LASSO

- "Least Absolute Shrinkage & Selection Operator"
- Problem: regression with many more features than examples $m << n$

  $\min_w \Sigma_i (y_i - x_i'w)^2$

  $\mathbb{R}^n$

  m examples
- Problem:

  overfitting

LASSO

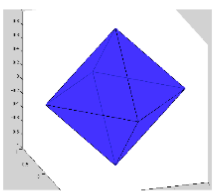
$$\min \sum_i (y_i - x_i \cdot w)^2 + \sum_j |w_j|$$

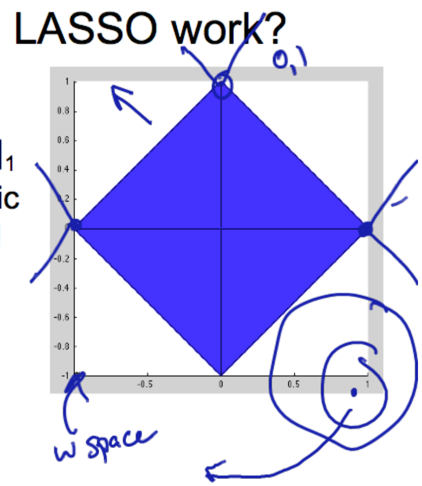
m exs

n weights

Why would LASSO work?

- Imagine constraining $||w||_1$
- Pretend quadratic is near-spherical





LASSO as QP

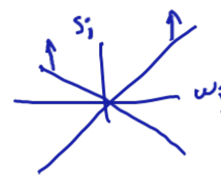
- Just like absolute value LP

$$\min_w \Sigma_i (y_i - x_i'w)^2 + \Sigma_j s_j$$

represents $|\cdot|$

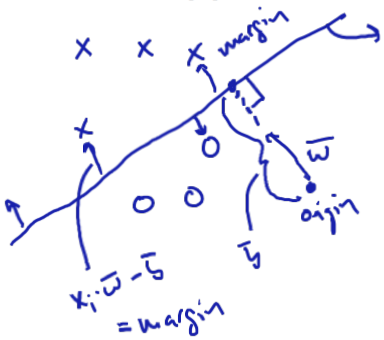
$$s_j \geq w_j$$
$$s_j \geq -w_j$$

$s_j \geq |w_j|$



Support vector machines

$\mathbb{R}^n$ points x_i $-1,+1$ y_i



$x_i \cdot w - b \geq 0$

$x_i \cdot \bar{w} - \bar{b} \geq 0$

Def

$\bar{w} = w/\|w\|$

$\bar{b} = b/\|w\|$

Maximizing margin

- margin = $y_i (x_i \cdot \bar{w} - \bar{b})$
- max M s.t. $y_i(x_i \cdot \bar{w} - \bar{b}) \geq M \ \forall i$