# Linear Programming: problem statement the geometry of LPs

Optimization - 10725

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January 16th, 2008





## Maximizing revenue

$\sum_{i=1}^{n} p_i x_i = p'x$

- n products, how much do we produce of each?
  - Amounts: $x_1, \ldots, x_n$
  - Profit for each product: $p_1, \ldots, p_n$
- m resources, quantities: $r_1, \ldots, r_m$
- Each product uses a certain amount of each resource:
- What's the optimal amount of each product?

profit $ for product i: $x_i \cdot p_i$

product i uses $a_{ij}$ of resource j per unit

objective function: $\max_{x_1, \ldots, x_n} \sum_{i=1}^{n} p_i x_i$ (linear)

Constraints: (subject to)

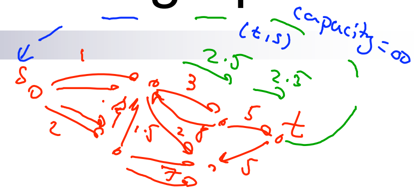
$\sum_i a_{ij} x_i \leq r_j \quad \forall j$ (linear)

$x_i \geq 0 \quad \forall i$

a linear program

# Maximum flow, directed graph

- Given a set of one-way streets
  - directed graph
  - Edges (i,j) ∈ E
  - Each edge has a capacity $c_{ij}$




- How much traffic can flow from source to destination?

Variables $f_{ij}$ for every edge $(i,j)$, flow from i to j along edge $(i,j)$

Objective: $\max_f \; f_{ts}$

Constraints: conservation of flow $\sum_i f_{ij} = \sum_k f_{jk} \quad \forall j$

$f_{ij} \le c_{ij}$ capacity constraint

$f_{ij} \ge 0 \quad \forall (i,j) \in (E \cup (t,s))$




# $L_1$ Regression

- Least-squares regression: $\mathbf{w}^* = \arg\min_{\mathbf{w}} \sum_j \left( t(\mathbf{x}_j) - \sum_i w_i h_i(\mathbf{x}_j) \right)^2$
- $L_1$ regression:
- Absolute values, not linear… how can this be solved???

# LPs are cool

- Fourier talked about LPs
- Dantzig discovered the Simplex algorithm in 1947
  - Exponential time
  - Versions of simplex are still among fastest LP solvers
- Many thought LPs were NP-hard…
- First polytime algorithm:
  - Khachiyan1979, first practical Karmarkar 1984
- Considered "hardest" polytime problem
- Many, many, many, many, many important practical apps
- Can approximate convex problems
- Basis for many, many, many, many approximation algorithms
- All in all, LPs are the foundation of "everything optimization", if you understand LPs, you are set to understanding the rest



# Graphical representation of LPs

- Constraints:
  - $x_1+2x_2 \leq 3$
  - $2x_1+x_2 \leq 3$
  - $x_1 \geq 0$, $x_2 \geq 0$
- Objective functions:

# Visualizing Solution - Intuition

- Dropping a ball




# Unbounded LPs - Intuition

- Where's my ball???

## Infeasible LPs - Intuition

- No room for the ball…




## General Form of Linear Program

- Variables:
- Objective:
- Constraints:
  - Linear combination of variables, constant coefficients
  - Less than:
  - Greater than:
  - Equality:
- Types of variables
  - Positive
  - Negative
  - Free

# Matrix Form

- What we have seen thus far:

- Matrix form:

# LPs in Standard Form

- **Standard form**:
  - Only equality and positivity constraints
  - Every LP can be written this way!
- Turning **inequalities into equalities**

- What about variables?
  - **Negative variables**:
  - **Free variables**:
- Side note, can also turn equality into inequalities, how?

# Understanding the Standard Form

- Standard form:
  - **Assuming all rows are linearly independent**
- **Fully-constrained** problem
  - n=m
  -
- **Over-constrained** problem
  - n<m
  -
- **Under-constrained** problem
  - n>m
  -




# Visualizing standard form LPs

- Standard form:
  - n>m
  - Hard to visualize
  - Corresponds to n-m linear subspace
- A very important example:

# Feasibility Problems

- "*Find a (any) vector that satisfies the constraints, or say it's infeasible*"
  - i.e., no objective
- Much easier?
  - Not quite…
- **Suppose you know optimal objective is v***
  - Solve feasibility problem
- **Don't know v***?




# Announcements

- Update about waiting list and auditors - Next week
  - Priority for people taking the class for credit
- Linear algebra review
  - Today, special TIME & LOCATION: 6-7PM NSH 1305
- First recitation, linear programming
  - Thursday, 5:00-6:20, Wean Hall 5409

# Understanding the Geometry of LPs

- Rest of today and part of next lecture: Understanding geometry of LPs
- Focus on inequality constraints, but works with equalities too
  - A few hints along the way
- Provides the foundation for
  - LP formulations
  - Duality
  - Solution methods
  - Conquering the world




# The Polyhedron*

- Definition:
  - **Inequality constraints**
  - (Can also contain equalities)
- Visualization

* Sometimes called polytope, noboby can agree on the definition

# Another view of polyhedra:
## Intersection of Halfspaces & Hyperplanes

- Half space:
- Hyperplane:
- Intersection:




# Infeasible LPs

- **LP is infeasible if and only if polyhedron defined by constraints is empty**
  - Feasibility doesn't depend on the objective function
- Another interesting case: Polyhedron is a point

## Convex Sets

- Definition:
  - "Every line segment between two points is in the set"
  - 
- Examples:




## Intersection of Convex Sets

- Fundamental Theorem:
  ***Intersection of convex sets is convex***
- What can we say about polyhedra?

# Interesting Case: Convex Hull

- A **convex combination**
- **Convex hull**
  - Set of all possible convex combinations
- Interesting fact: "*Given set of points in a convex set, their convex hull is contained in the convex set*"

# Extreme Points of a Polyhedron

- Extreme points cannot be represented as a linear combination of two other points in polyhedron
  -
- Examples:

# What you need to know

- Formulating an LP
  - E.g., $L_1$ Regression
- Visualization of LPs
  - Solution, unboundedness, feasibility, standard form
- General and Standard Forms of Linear Programs
  - And transformations
- Feasibility problems
- The Polyhedron
- Convex sets
- Convex Hull
- Extreme Points