# Linear Programming: the geometry of LPs

Optimization - 10725
Carlos Guestrin
Carnegie Mellon University

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## Understanding the Geometry of LPs

- Today's lecture: Understanding geometry of LPs
- Focus on inequality constraints, but works with equalities too
  - A few hints along the way
- Provides the foundation for
  - LP formulations
  - Duality
  - Solution methods
  - Conquering the world

# The Polyhedron*

- Definition:
  - **Inequality constraints**
  - (Can also contain equalities)

- Visualization

* Sometimes called polytope, noboby can agree on the definition




# Another view of polyhedra:
## Intersection of Halfspaces & Hyperplanes

- Half space:

- Hyperplane:

- Intersection:

# Infeasible LPs

- **LP is infeasible if and only if polyhedron defined by constraints is empty**
  - Feasibility doesn't depend on the objective function

- Another interesting case: Polyhedron is a point




# Convex Sets

- Definition:
  - "Every line segment between two points is in the set"
  - 

- Examples:

## Intersection of Convex Sets

- Fundamental Theorem:
  ***Intersection of convex sets is convex***

- What can we say about polyhedra?




## Interesting Case: Convex Hull

- A **convex combination**

- **Convex hull**
  - Set of all possible convex combinations

- Interesting fact: "*Given set of points in a convex set, their convex hull is contained in the convex set*"

## Extreme Points of a Polyhedron

- Extreme points cannot be represented as a linear combination of two other points in polyhedron
  - 
- Examples:




## Intuition about extreme points

- An extreme point for a polyhedron in $R^n$ is:
  - A feasible point
  - The unique intersection of n linearly separable hyperplanes

## Active constraints

- Given an LP
  - E.g.,
- An inequality constraint is **active** at a point $x^*$ if the constraint **holds with equality**
  - 
- BTW. If $x^*$ is a feasible point, then the equality constraints will always be active

 

## Basic solutions

- Consider a polytope:
- Given a set of n linearly independent active constraints
  - 
- **Basic solution**: unique solution for the resulting linear system of linearly independent constraints
  - 
- **Basic feasible solution**: a basic solution that satisfies *all constraints*
- BTW. In standard form, a basic feasible solution:
  - Satisfies m equality constraints, and
  - n-m inequality constraints

 

## Existence of basic feasible solutions

- Consider a polyhedron P
  - 
  - When does a basic feasible solution exist?
- Theorem: If polyhedron is not empty, and there are at least n linearly independent constraints, then there exists at least one basic feasible solution




## What can we do with basic (feasible) solutions?

- Suppose you know which constraints are active at the optimal point, then:
  - finding optimal solution is just matrix inversion
  - Solve LP by searching over active constraints
    - Basis of famous and effective (and worst case exponential) simplex algorithm
- How many basic (feasible) solutions?
  - Every subset of n linearly independent constraints could be a basic solution
  - Finite set!
  - Worst case?

## Basic feasible solutions and Extreme points

- Basic feasible solution $x^*$:
  - Feasible point
  - Unique solution to n linearly independent
- Extreme point $x^*$:
  - Cannot be written as a linear combination of other points
- Definitions are quite different
- Theorem: $x^*$ is a basic feasible solution if and only if $x^*$ is an extreme point



## Announcements

- If you are:
  - On the waiting list, or
  - Want to switch to audit
  - Sign list (again)
- Recitation, linear programming geometry
  - Thursday, 5:00-6:20, Wean Hall 5409
- Homework:
  - Out today
  - Due Monday Feb. 11, beginning of class

# Vertices of a polyhedron

- A vertex x of a polyhedron P
  - A point in P that is optimal for some objective function
  - Brings objective function back into the game!
- Formally, x is a vertex of P, if
  - x is in P
  - There exists a cost vector c, such that
    - Cost of x is lower than all other point y in P




# Vertices, extreme points and basic feasible solutions…

- Extreme points:
  - 
- Basic feasible solutions:
  - 
- Vertices:
  - 
- Very different…
- Theorem:
  - Proof:
  - E.g., $x^*$ vertex ⇒ $x^*$ extreme point
    - By definition, if $x^*$ is a vertex:
    - Assume $x^*$ is not an extreme point, then there exists y, z and λ.
    - Since $x^*$ is a vertex:
      - Thus:

## Vertices and Optimal Solutions

- LP problem:
- For every vertex $x^*$, there is a cost vector c
  - $x^*$ is optimal for c
- What about the other way?
  - For every cost vector (every LP), does there exist a vertex?




## Optimality of extreme points

- LP:
- If P
  - has at least one extreme point, and
  - there exists an optimal solution
  - *then there exists an optimal solution which is an extreme point of P*
- Proof:
  - Optimal value v:
  - Set of optimal solutions Q:
  - Q has extreme points:
  - $x^*$ is an extreme point of Q, then $x^*$ is an extreme point of P

- There are more general results in the readings

# What you need to know

- The Polyhedron
- Convex sets
- Convex Hull
- Extreme Points
- Active constraints
- Basic (feasible) solutions
- Vertices of a polyhedron
  - Brings objective function back into the game!
- Vertices, extreme points and basic feasible solutions: Equivalence
- Optimality of extreme points