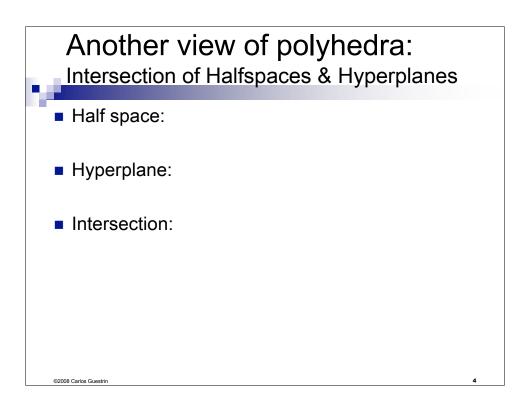


Understanding the Geometry of LPs

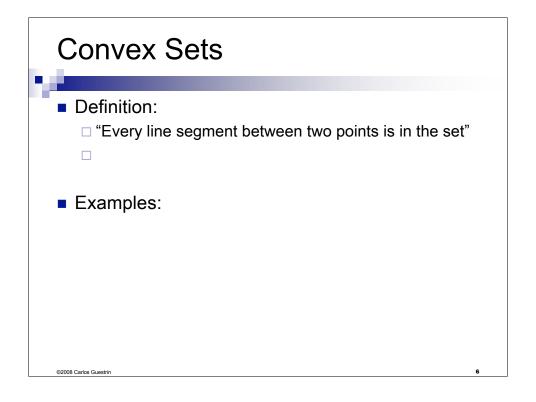
- - Today's lecture: Understanding geometry of LPs
 - Focus on inequality constraints, but works with equalities too
 - $\hfill \square$ A few hints along the way
 - Provides the foundation for
 - □ LP formulations
 - □ Duality
 - □ Solution methods
 - □ Conquering the world

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The Polyhedron* Definition: Inequality constraints Can also contain equalities) Visualization Sometimes called polytope, noboby can agree on the definition



Infeasible LPs LP is infeasible if and only if polyhedron defined by constraints is empty Feasibility doesn't depend on the objective function Another interesting case: Polyhedron is a point



Intersection of Convex Sets



Fundamental Theorem:
Intersection of convex sets is convex

What can we say about polyhedra?

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Interesting Case: Convex Hull



- A convex combination
- Convex hull
 - □ Set of all possible convex combinations
- Interesting fact: "Given set of points in a convex set, their convex hull is contained in the convex set"

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Extreme Points of a Polyhedron



 Extreme points cannot be represented as a linear combination of two other points in polyhedron

■ Examples:

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Intuition about extreme points



- An extreme point for a polyhedron in Rⁿ is:
 - □ A feasible point
 - □ The unique intersection of n linearly separable hyperplanes

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Active constraints



- Given an LP
 - □ E.g.,
- An inequality constraint is active at a point x^{*} if the constraint holds with equality

BTW. If x* is a feasible point, then the equality constraints will always be active

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Basic solutions



- Consider a polytope:
- Given a set of n linearly independent active constraints

 Basic solution: unique solution for the resulting linear system of linearly independent constraints

- Basic feasible solution: a basic solution that satisfies all constraints
- BTW. In standard form, a basic feasible solution:
 - □ Satisfies m equality constraints, and
 - □ n-m inequality constraints

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Existence of basic feasible solutions



Consider a polyhedron P

□ When does a basic feasible solution exist?

 Theorem: If polyhedron is not empty, and there are at least n linearly independent constraints, then there exists at least one basic feasible solution

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What can we do with basic (feasible) solutions?

- Suppose you know which constraints are active at the optimal point, then:
 - □ finding optimal solution is just matrix inversion
 - □ Solve LP by searching over active constraints
 - Basis of famous and effective (and worst case exponential) simplex algorithm
- How many basic (feasible) solutions?
 - □ Every subset of n linearly independent constraints could be a basic solution
 - □ Finite set!
 - □ Worst case?

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Basic feasible solutions and Extreme points

- Basic feasible solution x*:
 - □ Feasible point
 - □ Unique solution to n linearly independent
- Extreme point x*:
 - ☐ Cannot be written as a linear combination of other points
- Definitions are quite different
- Theorem: x* is a basic feasible solution if and only if x* is an extreme point

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Announcements



- If you are:
 - □ On the waiting list, or
 - □ Want to switch to audit
 - ☐ Sign list (again)
- Recitation, linear programming geometry
 - ☐ Thursday, 5:00-6:20, Wean Hall 5409
- Homework:
 - □ Out today
 - □ Due Monday Feb. 11, beginning of class

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Vertices of a polyhedron

- A vertex x of a polyhedron P
 - □ A point in P that is optimal for some objective function
 - □ Brings objective function back into the game!
- Formally, x is a vertex of P, if
 - □ x is in P
 - ☐ There exists a cost vector c, such that
 - Cost of x is lower than all other point y in P

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Vertices, extreme points and basic feasible solutions...



- Extreme points:
- Basic feasible solutions:
- Vertices:
- Very different...
- Theorem:
 - Proof:
 - \Box E.g., x^* vertex $\Rightarrow x^*$ extreme point
 - By definition, if x* is a vertex:
 - Assume x^* is not an extreme point, then there exists y, z and λ
 - Since x* is a vertex:

Vertices and Optimal Solutions



- LP problem:
- For every vertex x*, there is a cost vector c
 - □ x* is optimal for c
- What about the other way?
 - □ For every cost vector (every LP), does there exist a vertex?

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Optimality of extreme points



- IP.
- If P
 - $\hfill \square$ has at least one extreme point, and
 - □ there exists an optimal solution
 - $\hfill\Box$ then there exists an optimal solution which is an extreme point of P
- Proof:
 - Optimal value v:
 - Set of optimal solutions Q:
 - Q has extreme points:
- There are more general results in the readings

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What you need to know



- The Polyhedron
- Convex sets
- Convex Hull
- Extreme Points
- Active constraints
- Basic (feasible) solutions
- Vertices of a polyhedron
 - □ Brings objective function back into the game!
- Vertices, extreme points and basic feasible solutions: Equivalence
- Optimality of extreme points

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