Linear Programming: the geometry of LPs

Optimization - 10725
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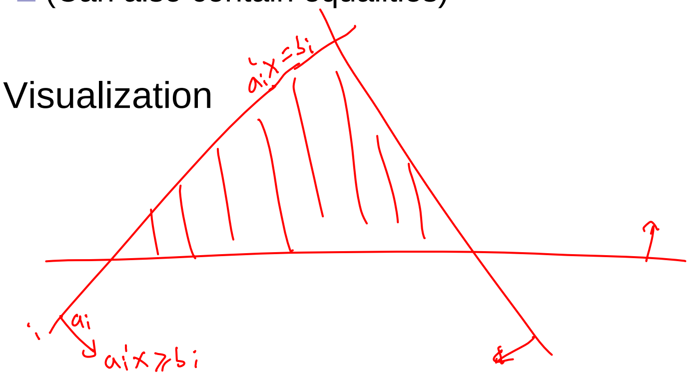
Understanding the Geometry of LPs

- Today's lecture: Understanding geometry of LPs
- Focus on inequality constraints, but works with equalities too
  - A few hints along the way
- Provides the foundation for
  - LP formulations
  - Duality
  - Solution methods
  - Conquering the world

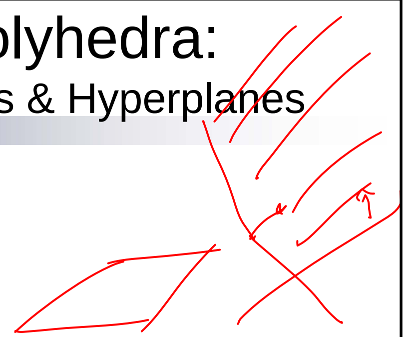
The Polyhedron*

- Definition: $P = \{x \mid Ax \geq b\}$
  - **Inequality constraints** $\forall i \;\; a_i x \geq b_i$
  - (Can also contain equalities)
- Visualization



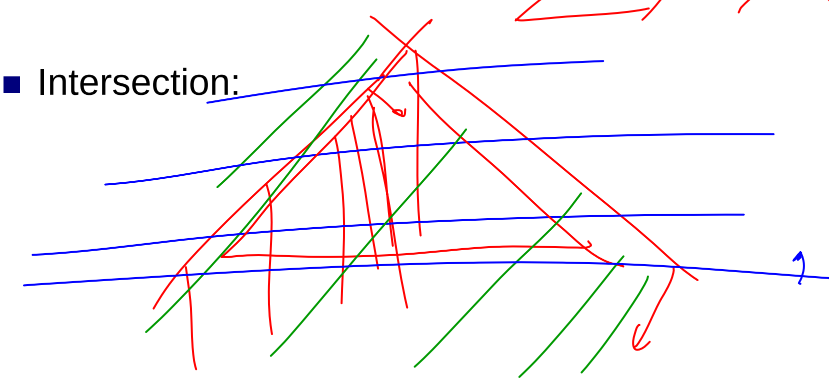
* Sometimes called polytope, noboby can agree on the definition

Another view of polyhedra:
Intersection of Halfspaces & Hyperplanes



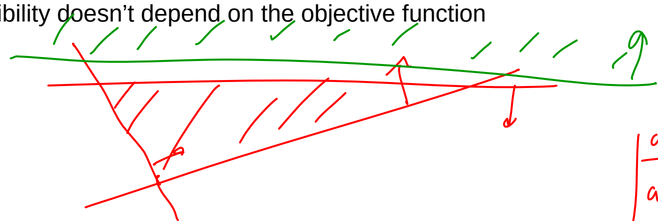
- Half space:

- Hyperplane: $a_i x = b_i$
- Intersection:



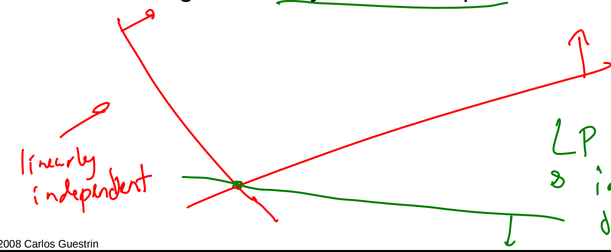
Infeasible LPs

- **LP is infeasible if and only if polyhedron defined by constraints is empty**
  - Feasibility doesn't depend on the objective function



a_1 & a_2 are linearly dependent

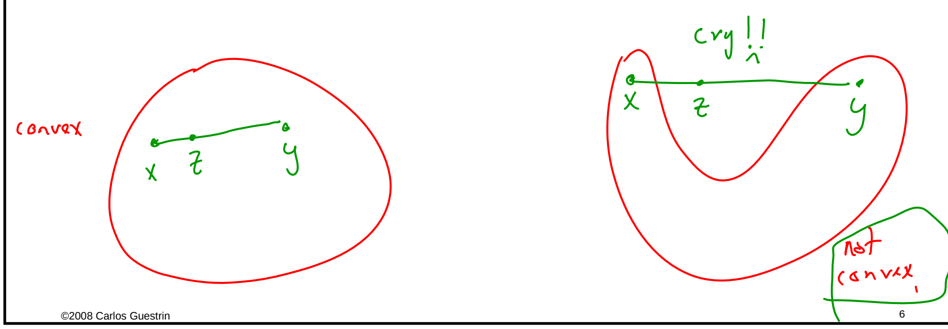
- Another interesting case: Polyhedron is a point



LP has 1 solution & independent of objective function

Convex Sets

- Definition: Convexity
  - "Every line segment between two points is in the set"
  - $z = \lambda x + (1-\lambda) y \quad \lambda \in [0,1]$
    if $x \in P$, $y \in P \Rightarrow z \in P$
- Examples:

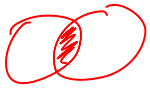


Intersection of Convex Sets

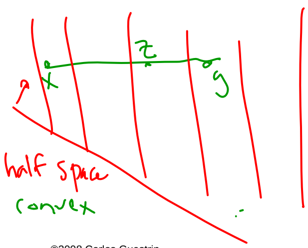
- Fundamental Theorem:
  Intersection of convex sets is convex

P_i convex $\forall i$ $\Rightarrow$ if $P = \cap_i P_i$

P is convex



- What can we say about polyhedra?



Polyhedra convex,

because intersections of half-spaces

Interesting Case: Convex Hull

- A **convex combination**

λ_i $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$

$x = \sum \lambda_i x_i$

0.4 y

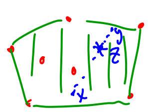
x 0.5

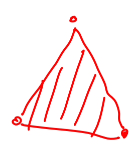
z

weighted avg.

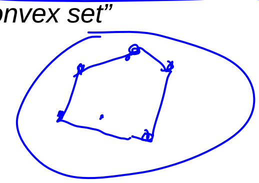
- **Convex hull**
  - Set of all possible convex combinations

polyhedra convex



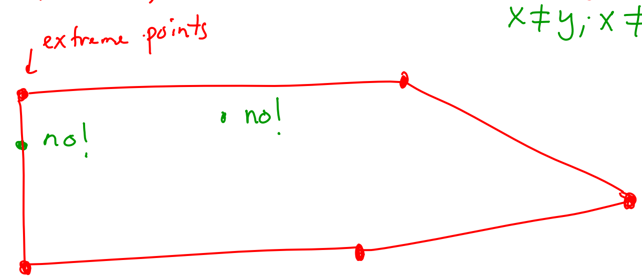


- Interesting fact: "*Given set of points in a convex set, their convex hull is contained in the convex set*"



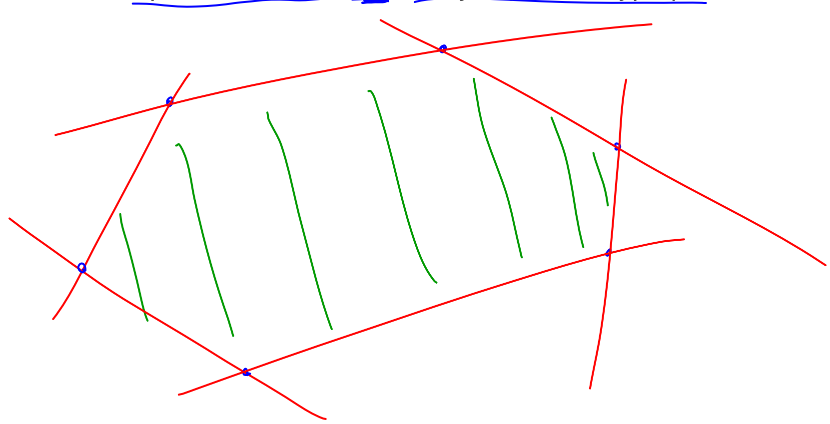
Extreme Points of a Polyhedron

- Extreme points cannot be represented as a convex ~~linear~~ combination of two other points in polyhedron
  - $x \in P$ is an extreme point, if $\nexists\, y \in P, z \in P, \lambda \in [0,1]$, such that $x = \lambda y + (1-\lambda) z$, $x \neq y, x \neq z$
- Examples:



Intuition about extreme points

- An extreme point for a polyhedron in R^n is:
  - A feasible point
  - The unique intersection of n linearly ~~separable~~ independent hyperplanes

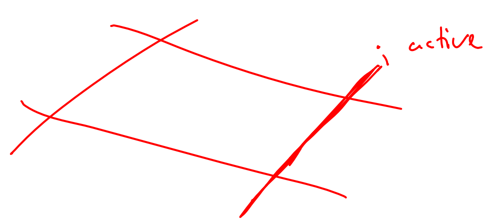


Linearly Independent

Active constraints

- Given an LP $\min_x c'x$
  - E.g., $Ax \geq b \equiv a_i x \geq b_i \ \forall i$

- An inequality constraint is **active** at a point x^* if the constraint **holds with equality**
  - $a_i x^* = b_i \Rightarrow i$ is active

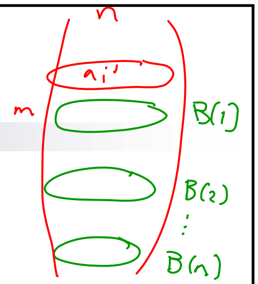


- BTW. If x^* is a feasible point, then the equality constraints will always be active $Ax^* = b$

Basic solutions

$x \in \mathbb{R}^n$

m constraints

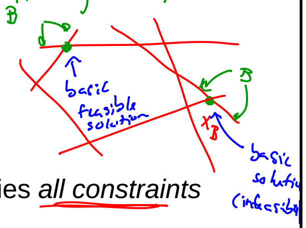


- Consider a polytope: $P = \{x \mid Ax \geq b\}$

  $n \leq m$

B forms a basis

- Given a set of n linearly independent active constraints
  - $B \subset \{1,\dots,m\}$ $|B| = n$; vectors $a_{B(i)}$, $i = 1..n$, are linearly independent
- **Basic solution**: unique solution for the resulting linear system of linearly independent constraints
  - $A_B = \begin{pmatrix} a_{B(1)}' \\ \vdots \\ a_{B(i)}' \\ \vdots \\ a_{B(n)}' \end{pmatrix}$ n invertible $b_B = \begin{pmatrix} b_{B(1)} \\ b_{B(i)} \\ \vdots \\ b_{B(n)} \end{pmatrix}$ $A_B x_B = b_B$ $x_B = A_B^{-1} b_B$



- **Basic feasible solution**: a basic solution that satisfies *all constraints*
- BTW. In standard form, a basic feasible solution:
  - Satisfies m equality constraints, and
  - n-m inequality constraints are active

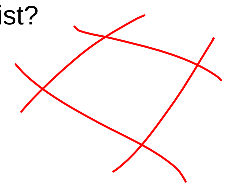
$\min_x x$, $Ax = b$, $x \geq 0$, $n \geq m$

$n-m$ variables are $\emptyset$

Existence of basic feasible solutions

- Consider a polyhedron P
  - $Ax \geq b$
  - When does a basic feasible solution exist?

$\leftarrow$ no basic feasible solutions



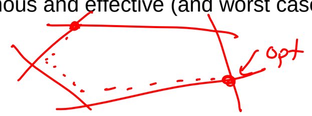
- Theorem: If polyhedron is not empty, and there are at least n linearly independent constraints, then there exists at least one basic feasible solution

What can we do with basic (feasible) solutions?

assuming B^* has n LI vectors

- Suppose you know which constraints are active (B^*) at the optimal point (x^*), then:
  - finding optimal solution is just matrix inversion

    $x^* = A_{B^*}^{-1} b_{B^*}$
  - Solve LP by searching over active constraints
    - Basis of famous and effective (and worst case exponential) simplex algorithm



- How many basic (feasible) solutions?
  - Every subset of n linearly independent constraints could be a basic solution
  - Finite set!
  - Worst case? $\binom{m}{n}$

$$\min_x c'x$$
$$_m \overset{n}{A} x \geq b$$

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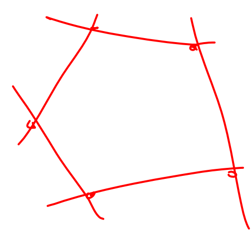
Basic feasible solutions and Extreme points

- Basic feasible solution x^*:
  - Feasible point
  - Unique solution to n linearly independent

    $A_B x^* = b_B$



- Extreme point x^*:
  - Cannot be written as a linear combination of other points

    $\nexists\, y, z, \lambda \,,\quad x^* = \lambda y + (1-\lambda) z$

- Definitions are quite different
- Theorem: x^* is a basic feasible solution if and only if x^* is an extreme point

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Announcements

Reading.

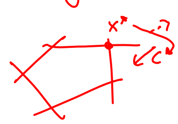
- If you are:
  - On the waiting list, or
  - Want to switch to audit
  - Sign list (again)
- Recitation, linear programming geometry
  - Thursday, 5:00-6:20, Wean Hall 5409
- Homework:
  - Out today
  - Due Monday Feb. 11, beginning of class

Vertices of a polyhedron

$\min_x c'x$
$Ax \geq b$

- A vertex x^* of a polyhedron P
  - A point in P that is optimal (uniquely) for some objective function c

$\exists c$, s.t. $c'x^* < c'y$ $x^* \in P$, $\forall y \in P$ $y \neq x^*$



  - Brings objective function back into the game!
- Formally, x is a vertex of P, if
  - x is in P
  - There exists a cost vector c, such that
    - Cost of x is (strictly) lower than all other point y in P

$\exists c$ st. $c'x^* < c'y$ $\forall y \in P$, $y \neq x^*$

Vertices, extreme points and basic feasible solutions…

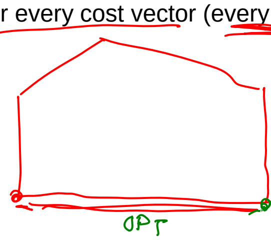
- Extreme points:
  - $\nexists\ \lambda, y, z \quad x = \lambda y + (1-\lambda) z$
- Basic feasible solutions:
  - $A_B x_B = b_B$
- Vertices:
  - $\exists c \quad c'x < c'y \quad \forall y \in P,\ y \neq x$
- Very different…
- Theorem: All equivalent!
  - Proof: in the reading
  - E.g., x^* vertex $\Rightarrow$ x^* extreme point
    - By definition, if x^* is a vertex: $\exists c$ s.t. $c'x^* < c'y \quad \forall y \neq x^* \quad y \in P$
    - Assume x^* is not an extreme point, then there exists y, z and λ $\quad x^* = \lambda y + (1-\lambda) z$
    - Since x^* is a vertex: $\left.\begin{array}{l}\lambda\ c'y > c'x^* \\ (1-\lambda)\ c'z > c'x^*\end{array}\right\}$ $\quad \lambda c'y + (1-\lambda) c'z > c'x^*$
      - Thus: $c'[\lambda y + (1-\lambda) z] > c'x^*$

$x^* \neq \lambda y + (1-\lambda) z$

qed

Vertices and Optimal Solutions

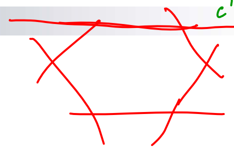
- LP problem: $\min_x \ c'x$
  $Ax \geq b$
- For every vertex x^*, there is a cost vector c
  - x^* is optimal for c
- What about the other way?
  - For every cost vector (every LP), does there exist a vertex?



not all solutions are vertices
but ∃ a vertex solution

Optimality of extreme points

- LP: $\min_x c'x$ $\quad Ax \geq b$



- If P $= \{x \mid Ax \geq b\}$
  - has at least one extreme point, and
  - there exists an optimal solution
  - *then there exists an optimal solution which is an extreme point of P*

- Proof:
  - Optimal value v: $v = c'x^*$
  - Set of optimal solutions Q: $Q = \{x \mid Ax \geq b,\ c'x = v\}$
  - Q has extreme points: Since P has extreme points, intersection with something polyhedron still has extreme points
  - x^* is an extreme point of Q, then x^* is an extreme point of P

    see reading...

- There are more general results in the readings

What you need to know

- The Polyhedron
- Convex sets
- Convex Hull
- Extreme Points
- Active constraints
- Basic (feasible) solutions
- Vertices of a polyhedron
  - Brings objective function back into the game!
- Vertices, extreme points and basic feasible solutions: Equivalence
- Optimality of extreme points