Linear Programming: problem statement the geometry of LPs

Optimization - 10725
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January 16th, 2008

Maximizing revenue

$\sum_{i=1}^{n} p_i x_i = p'x$

- n products, how much do we produce of each?
  - Amounts: $x_1, \ldots, x_n$
  - Profit for each product: $p_1, \ldots, p_n$
- m resources, quantities: $r_1, \ldots, r_m$
- Each product uses a certain amount of each resource:
- What's the optimal amount of each product?

profit for product i
$x_i \cdot p_i$

product i uses a_{ij} of resource j per unit

objective function: $\max_{x_1, \ldots, x_n} \sum_{i=1}^{n} p_i x_i$ linear

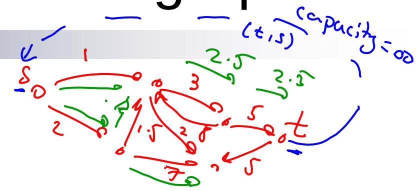
constraints: (subject to) $\sum_i a_{ij} x_i \leq r_j \quad \forall j$

$x_i \geq 0 \quad \forall i$

linear

a linear program

Maximum flow, directed graph



- Given a set of one-way streets
  - directed graph
  - Edges (i,j) $\in$ E
  - Each edge has a capacity c_{ij}
- How much traffic can flow from source to destination?

Variables f_{ij} for every edge (i,j), flow from i to j along edge (i,j)

Objective: $\max_f \; f_{ts}$

Constraints: conservation of flow

$$\sum_i f_{ij} = \sum_k f_{jk} \quad \forall j$$

$$f_{ij} \le c_{ij} \quad \text{capacity constraint}$$

$$f_{ij} \ge 0 \quad \forall (i,j) \in \left(E \cup (t,s)\right)$$

L_1 Regression

$\langle x_j, t(x_j) \rangle$ $\langle GPA, salary \rangle$
4.3 , 250K
⋮

- Least-squares regression: $\mathbf{w}^* = \arg\min_{\mathbf{w}} \sum_j \left(t(\mathbf{x}_j) - \sum_i w_i h_i(\mathbf{x}_j) \right)^2$

outlier, e.g. 4.45, 150K
error squared

- L_1 regression:

$$w^* = \arg\min_w \|t - Hw\|_1 = \arg\min_w \sum_j \left| t(x_j) - \sum_i w_i h_i(x_j) \right|$$

- Absolute values, not linear… how can this be solved???

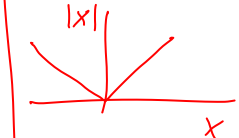
$$\min_{w,\varepsilon} \sum_j \varepsilon_j$$

constant for each x_j

$$\varepsilon_j \ge t(x_j) - \sum_i w_i h_i(x_j) \quad \forall j$$

$$\varepsilon_j \ge \left[\sum_i w_i h_i(x_j)\right] - t(x_j)$$

abs: $|x|$



$|a|$? $|-3|$

$\min_\varepsilon \varepsilon$

$\varepsilon \ge a$

$\varepsilon \ge -a$

LPs are cool

- Fourier talked about LPs
- Dantzig discovered the Simplex algorithm in 1947
  - Exponential time
  - Versions of simplex are still among fastest LP solvers
- Many thought LPs were NP-hard…
- First polytime algorithm:
  - Khachiyan1979, first practical Karmarkar 1984
- Considered "hardest" polytime problem
- Many, many, many, many, many important practical apps
- Can approximate convex problems
- Basis for many, many, many, many approximation algorithms
- All in all, LPs are the foundation of "everything optimization", if you understand LPs, you are set to understanding the rest

Graphical representation of LPs

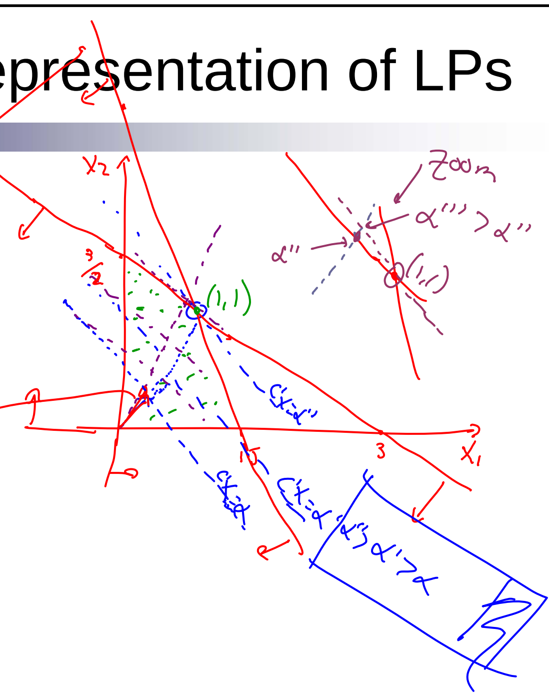
- Constraints:
  - $x_1+2x_2 \leq 3$
  - $2x_1+x_2 \leq 3$
  - $x_1 \geq 0,\ x_2 \geq 0$
- Objective functions:

max $x_1 + x_2 = (1,1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$x_1 = 1,\ x_2 = 1$ c

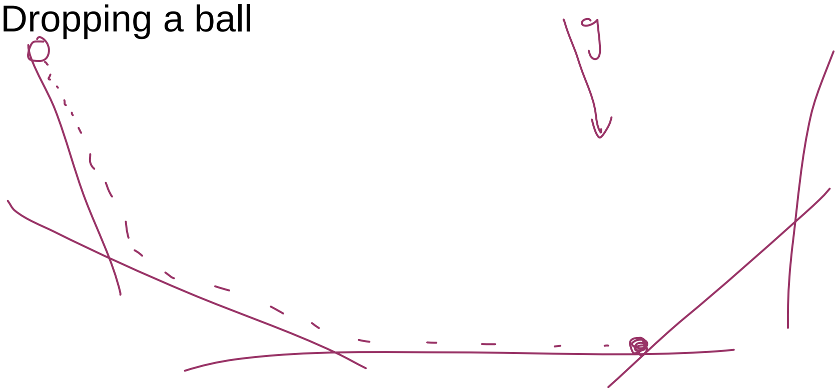
max $x_1 + 1.1x_2$

$x_1 = 1,\ x_2 = 1$



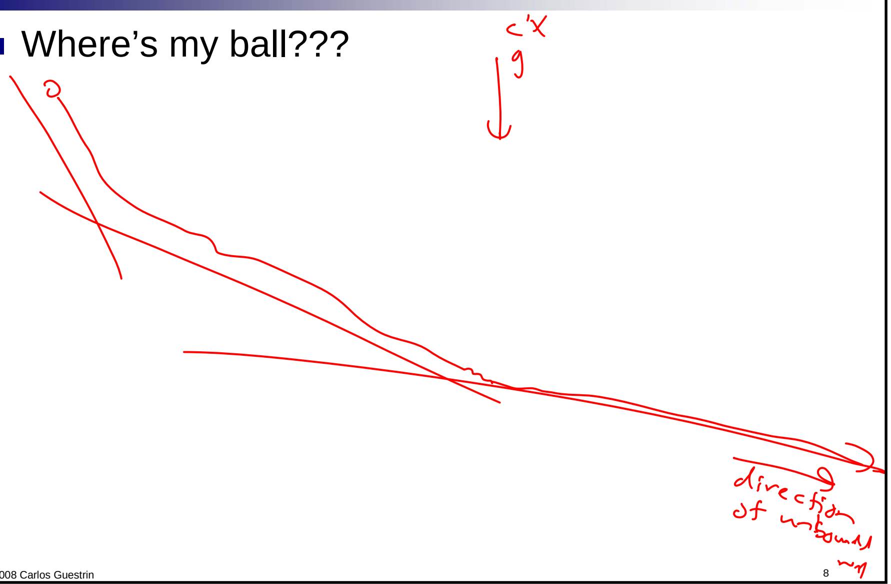
Visualizing Solution - Intuition

- Dropping a ball



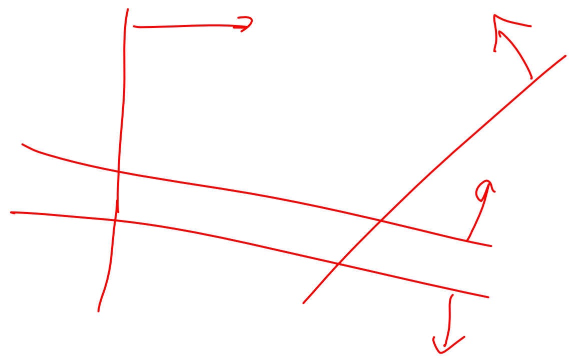
Unbounded LPs - Intuition

- Where's my ball???



Infeasible LPs - Intuition

- No room for the ball…



General Form of Linear Program

- Variables: $x_1, \dots, x_n$ ← want to optimize
- Objective: $\underset{\text{or max}}{\min} \sum_i c_i x_i = c'x$
- Constraints:
  - Linear combination of variables, constant coefficients
  - Less than: $a_i'x \le b_i$: $\sum_j a_{ij} x_j \le b_i$
  - Greater than: $a_i'x \ge b_i$
  - Equality: $a_i'x = b_i$
- Types of variables
  - Positive $x_j \ge 0$
  - Negative $x_j \le 0$
  - Free x_j free, $x_j \in \mathbb{R}$

$$\min_x c'x$$
$a_i'x \ge b_i \quad i \in M_1$
$a_i'x \le b_i \quad i \in M_2$
$a_i'x = b_i \quad i \in M_3$
$x_j \ge 0 \quad j \in N_1$
$x_j \le 0 \quad j \in N_2$
x_j free $\quad j \in N_3$

Matrix Form

- What we have seen thus far:

n vars
m constraints

$\min c'x$
$a_i x \geq b_i$
$x_j \geq 0$

- Matrix form:

$\min c'x$
$Ax \geq b$
$x \geq 0$

$A = \begin{array}{cc} & n \\ m & \begin{bmatrix} \quad a_i \quad \\ \\ \\ \end{bmatrix} \end{array}$ $b = \begin{array}{cc} & 1 \\ m & \begin{bmatrix} b_i \\ \\ \\ \end{bmatrix} \end{array}$

LPs in Standard Form

if $\max_x c'x$
then $\min_x -c'x$

- **Standard form**:
  - Only equality and positivity constraints
  - Every LP can be written this way!

$\min_x c'x$
$Ax = b$
$x \geq 0$

- Turning **inequalities into equalities**

$a'x \geq b \iff a'x = b + \varepsilon$
$\varepsilon \geq 0$

- What about variables?
  - **Negative variables**: $x \leq 0 \rightarrow$ change to $\bar{x}$, and negate all coeff.
  - **Free variables**: x free $\rightarrow$ substitute for, $x_1 \geq 0$, $x_2 \geq 0$, wherever x shows up, substitute for $x_1 - x_2$
- Side note, can also turn equality into inequalities, how?

$a'x = b \rightarrow a'x \geq b$ & $a'x \leq b$

Understanding the Standard Form

- Standard form:
  - Assuming all rows are linearly independent

$\min_x c'x$

$m\ \hat{A}x = b$

$x \geq 0$

- **Fully-constrained** problem
  - n=m
  - objective is irrelevant !!

$Ax = b$ has unique solution x^*

$x^* \geq 0$, then x^* opt

x^* has some component < 0, then infeasible

- **Over-constrained** problem
  - n<m
  - can't be linearly indep (Booooogus) if we have n linearly indep. constraints
- **Under-constrained** problem
  - n>m
  - room to optimize $c'x$

    & LPs are cool!! :)

Visualizing standard form LPs

- Standard form:

$\min c'x$

$Ax = b$

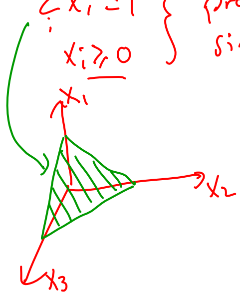
$x \geq 0$

  - n>m
  - Hard to visualize
  - Corresponds to n-m linear subspace

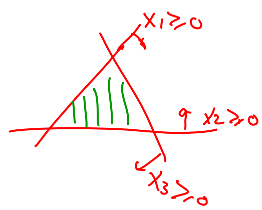
if linearly independent constraints

- A very important example:

$\sum x_i = 1$, $x_i \geq 0$ } probability simplex



walk on the n-m subspace



Feasibility Problems

- "*Find a (any) vector that satisfies the constraints, or say it's infeasible*"
  - i.e., no objective

give me an x such that $Ax \geq b$

- Much easier?
  - Not quite…

min $c'x$
$Ax \geq b$

- **Suppose you know optimal objective is v*** $= c'x^*$
  - Solve feasibility problem

give me an x such that $Ax \geq b$, $c'x \leq v^*$

- **Don't know v***?

Suppose I know $v^* \in [L, R]$

Binary search

$\hat{v}_0 = \frac{L+R}{2}$

solve $Ax \geq b$, $c'x \leq v_i$ for feasibility
- feasible $\to$ v_i is too large
- infeasible $\to$ v_i is too small

Announcements

- Update about waiting list and auditors - Next week
  - Priority for people taking the class for credit
- Linear algebra review
  - Today, special TIME & LOCATION: 6-7PM NSH 1305
- First recitation, linear programming
  - Thursday, 5:00-6:20, Wean Hall 5409

Office hours start next week

Understanding the Geometry of LPs

- Rest of today and part of next lecture: Understanding geometry of LPs

- Focus on inequality constraints, but works with equalities too
  - A few hints along the way

- Provides the foundation for
  - LP formulations
  - Duality
  - Solution methods
  - Conquering the world

The Polyhedron*

- Definition:
  - **Inequality constraints**
  - (Can also contain equalities)

- Visualization

* Sometimes called polytope, noboby can agree on the definition

Another view of polyhedra:
Intersection of Halfspaces & Hyperplanes

- Half space:
- Hyperplane:
- Intersection:

Infeasible LPs

- **LP is infeasible if and only if polyhedron defined by constraints is empty**
  - Feasibility doesn't depend on the objective function
- Another interesting case: Polyhedron is a point

Convex Sets

- Definition:
  - "Every line segment between two points is in the set"
  -
- Examples:

Intersection of Convex Sets

- Fundamental Theorem:
  Intersection of convex sets is convex
- What can we say about polyhedra?

Interesting Case: Convex Hull

- A **convex combination**
- **Convex hull**
  - Set of all possible convex combinations
- Interesting fact: "*Given set of points in a convex set, their convex hull is contained in the convex set*"

Extreme Points of a Polyhedron

- Extreme points cannot be represented as a linear combination of two other points in polyhedron
  -
- Examples:

What you need to know

- Formulating an LP
  - E.g., L_1 Regression
- Visualization of LPs
  - Solution, unboundedness, feasibility, standard form
- General and Standard Forms of Linear Programs
  - And transformations
- Feasibility problems
- The Polyhedron
- Convex sets
- Convex Hull
- Extreme Points