LP duality cheat sheet

min c'x + d'y s.t.	max p'v + q'w s.t.
Ax + By ≥ p	A'v + E'w = c
Ex + Fy = q	B'v + F'w ≤ d
x free, y ≥ 0	v ≥ 0, w free
$\begin{pmatrix} A & B \\ E & F \end{pmatrix}$	$\begin{pmatrix} A^T & E^T \\ B^T & F^T \end{pmatrix}$

Swap RHS and objective
Swap max/min

Transpose constraint matrix
+ve vars yield ≤, free vars yield =

Linear feasibility problem

LP

min c'x s.t.
Ax + b ≥ 0

$c^T x^* \in [L, U]$

$G = \frac{L+U}{2}$

linear feasibility

find x s.t.
Ax + b ≥ 0

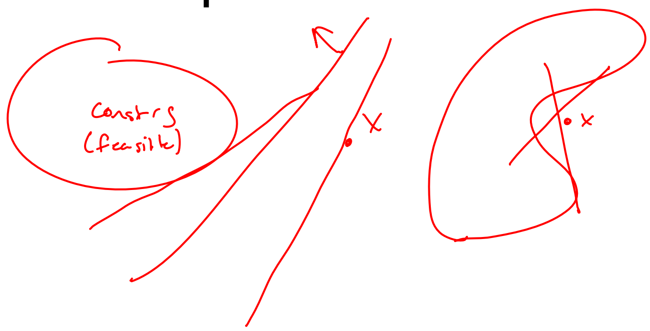
find x s.t.
$Ax + b \geq 0$
$c^T x \leq G$

feasible ⇒ [L, G]
infeasible ⇒ [G, U]

p(M) steps to solve
polynomial ↗ ; ↖ bit length of input

P(M) $\log_2(\frac{1}{\epsilon})$ steps
↖ relative accuracy

Separation oracle

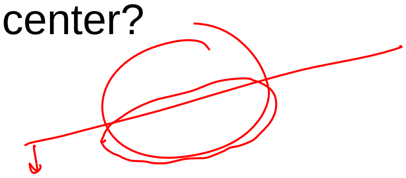


Ellipsoid preview

Difficulties

- How do we get bounding sphere? later
- How do we know when to stop?

  $\forall i \in I \quad a_i^T x + b_i + \eta \geq 0 \qquad \eta > 0$, small
- Bound region gets complicated–how do we find its center?



Bounding a partial ellipsoid

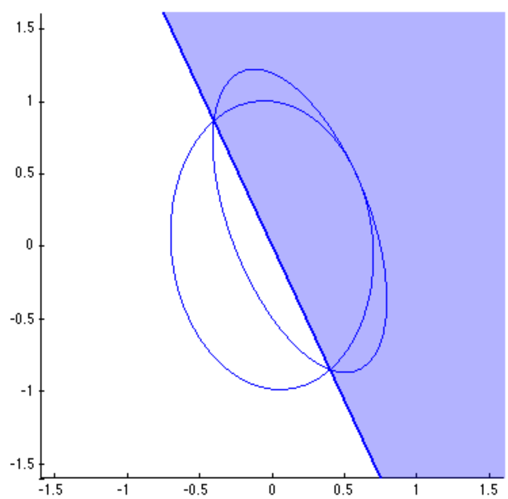
- General ellipsoid w/ center x_C, radius R:
- Halfspace: $p^T x + q \leq 0$
- Translate to origin, scale to be spherical

  y = x =

Bounding a partial sphere

- Rotate so hyperplane is axis-normal
- New center x_c:
- New shape A:

For example



Ellipsoid algorithm

- Want to find x s.t. $Ax+b+\eta \geq 0$
- Pick R s.t. $||x^*|| \leq R$
- $E_0 := \{ x \mid ||x|| \leq R \}$
- Repeat:
  - x_t := center of E_t
  - ask whether $Ax_t + b \geq 0$
    - yes: declare feasible!
    - no: get separating hyperplane
  - $E_{t+1} := \text{bound}(E_t \cap \{ x \mid p_t^T x \leq p_t^T x_t \})$
  - if $\text{vol}(E_{t+1}) \leq \varepsilon \text{vol}(E_0)$: declare infeasible!

Getting bounds

- How big do L, U need to be?
- How big does R need to be?
- What should η be?
- How small does ε need to be?

Other algorithms

- Ellipsoid is polynomial, but slow
- Some other algorithms:
  - simplex: exponential in worst case, but often fast in practice
  - randomized simplex: polynomial [Kelner & Spielman, 2006]
  - interior point: polynomial
  - subgradient descent: weakly polynomial, but really simple, and fast for some purposes

What's a subgradient?

Subgradient descent for SVMs

- $\min_{s,w,b}$ $||w||^2 + C\sum_i s_i$ s.t.
  $y_i(x_i^T w - b) \geq 1 - s_i$
  $s_i \geq 0$
- Equivalently,

Subgradient in SVM

- $\min_w L(w) = ||w||$^2 $+ C\sum_i h(y_i x_i^T w)$
- Subgradient of h(z):
- Subgradient of L(w) wrt w: