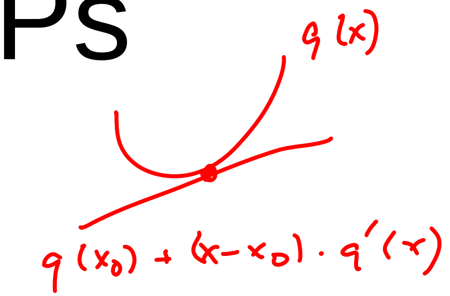
Solving convex programs

- Linear programs: simplex, subgradient, ellipsoid, interior point
- General CP: "simplex," subgradient, ellipsoid, interior point
- Interesting special cases: QP, SOCP, SDP

Separation oracle: QPs



- min $\underbrace{q(x)}_{\text{quadratic}}$ st $\underline{Ax = b}$, $\underline{x \geq 0}$

$\min_{z,x} \; z \quad \text{s.t.} \quad \underline{Ax = b} \quad \underline{x \geq 0} \quad \underline{\underline{z \geq q(x)}}$

$x_0, z_0 \quad \text{s.t.} \quad z_0 < q(x_0)$ ← violated by x_0

$z_0 < q(x_0) + (x - x_0) \cdot q'(x_0)$

$z \geq q(x_0) + \underline{(x - x_0) \cdot q'(x_0)}$ ← sat. by all feasible x, z

Separation oracle: SOCPs

- SOC constraint: $||Ax + b|| \leq c'x + d$
- Given x_0 that fails:

$$u = \frac{Ax_0 + b}{||Ax_0 + b||} \quad \text{or} \quad u = e_1 \quad \text{if } Ax_0 + b = 0$$

new constr: $u^T (Ax + b) \leq c^T x + d$

plug in x_0: $\underbrace{u^T (Ax_0 + b)}$ $\overset{?}{\leq} c^T x_0 + d \Rightarrow ||Ax_0 + b|| \overset{?}{\leq} c^T x_0 + d$

$\Rightarrow$ violated

$$= \frac{(Ax_0 + b)^T}{||Ax_0 + b||} (Ax_0 + b) = ||Ax_0 + b||$$

for any feasible x: $u^T (Ax + b) \leq \underset{}{\overset{=1}{\cancel{||u||}}} \, ||Ax + b|| \leq c^T x + d$

Separation oracle: SDPs

$A_i \in \mathbb{R}^{n \times n}$ $\quad A_i = A_i^T$

$x_i \in \mathbb{R}$

- SDP constraint:

  $A = x_1 A_1 + x_2 A_2 + \ldots$

  $A \in S_+$

given x_0, A_0 w/ $A_0 \not\succeq 0$ but $A = \sum_i x_i A_i$

$\Rightarrow \exists u \neq 0.$ $u^T A_0 u < 0$ $\Rightarrow$ A_0 violates $u^T A u \geq 0$

Convex duality

- Several new types of duality
  - convex cones
  - convex sets
  - convex functions
  - convex programs
- Generalize LP/QP duality
- Generalize norm duality (e.g., L_1 v. L_∞)

Cone duality

- Cone K (not necessarily convex)

$x \in K \Rightarrow \lambda x \in K \quad \lambda \geq 0$

- K* = $\{ y \mid x \cdot y \geq 0 \;\; \forall x \in K \}$

K^* = dual cone, polar cone

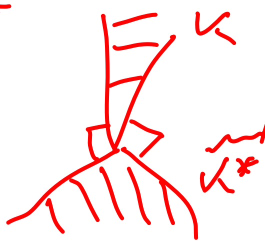
$\Leftrightarrow -y$ normal of halfspace containing K

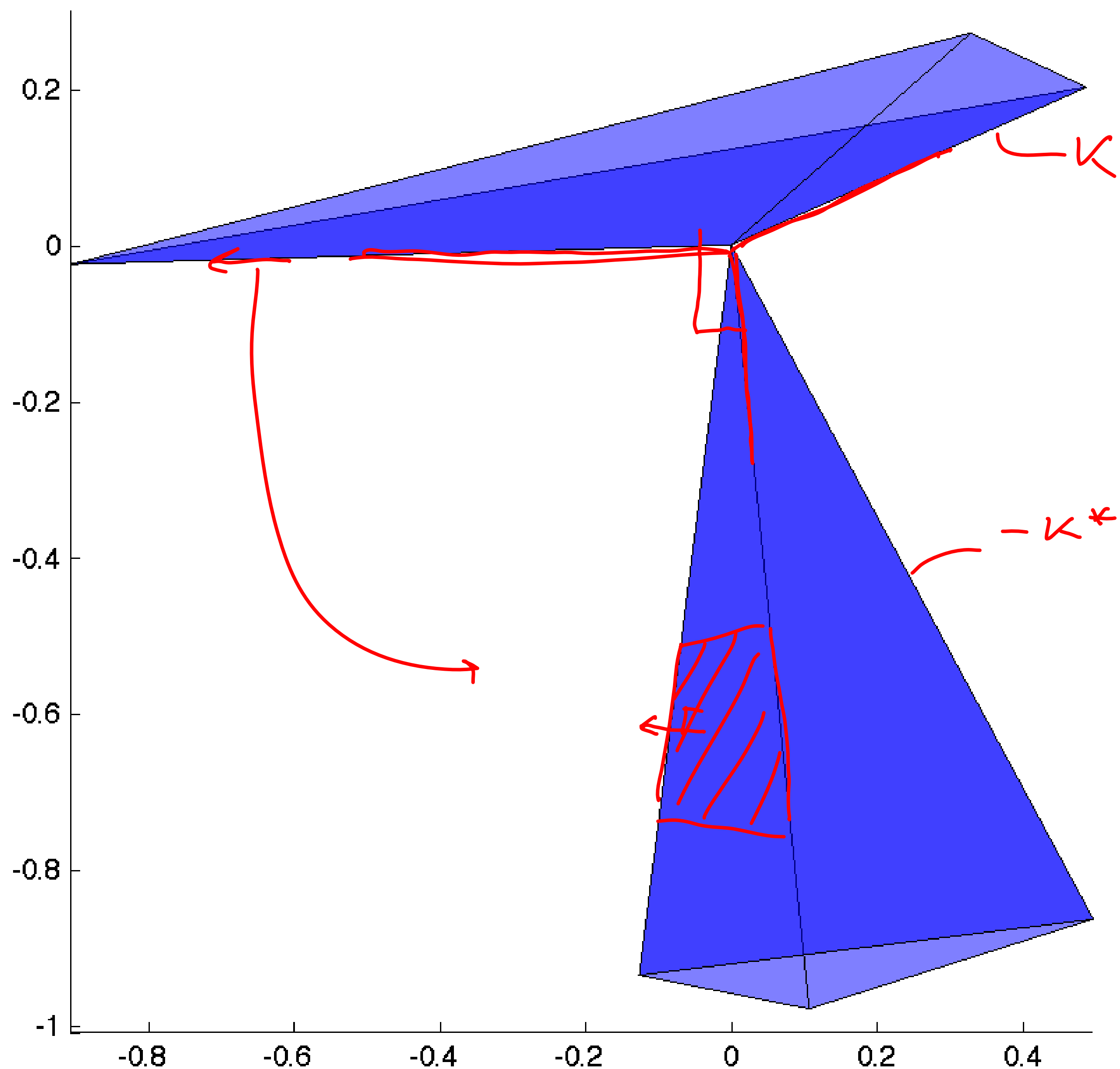
$x \cdot (-y) \leq 0$

Some books define K^*

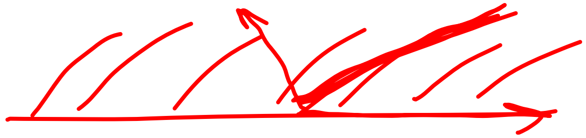
$= \{ y \mid x \cdot y \leq 0 \;\; \forall x \in K \}$

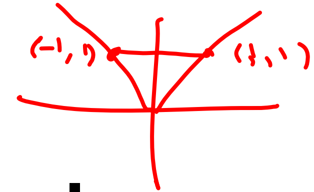
$\tilde{K}^* = -K^*$





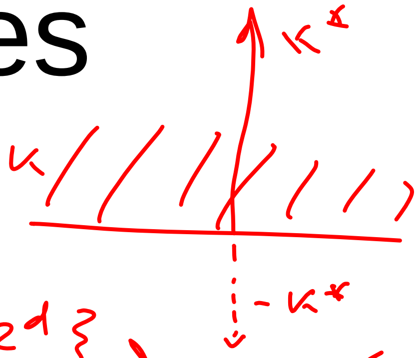
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
K
$-K^*$
x





Examples of dual cones

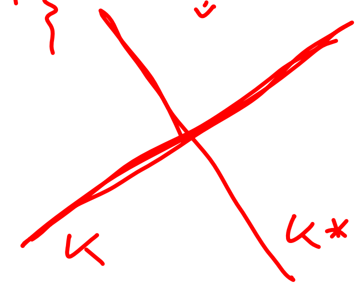
$\{0\} \Rightarrow \mathbb{R}^n$



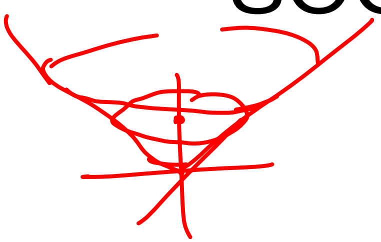
- Halfspace $a^T x \geq 0$ $\quad \{ \lambda a \mid \lambda \geq 0 \}$
- Subspace $\{ x \mid Ax = 0 \}$ $\quad \{ A^T y \mid y \in \mathbb{R}^d \}$
- $(R_+^n)^* = \mathbb{R}_+^n$

  $x \geq 0,\ y \geq 0 \Rightarrow x^T y \geq 0$

  $x = (3, -1, 2) \quad y = (0, 1, 0) \quad x^T y = -1$



- SOC: $\{ (x, s) \mid \|x\|_2 \leq s \}$ $\quad soc^* = soc$



$\begin{pmatrix} x \\ s \end{pmatrix} \cdot \begin{pmatrix} y \\ t \end{pmatrix} = x \cdot y + st \geq -\|x\| \|y\| + st \geq 0 \Leftarrow (\|x\| \|y\| \leq st)$

since x, s $\quad y, t$ $\in soc$

if $x = y$, $s = t = \|x\| \Rightarrow -s \cdot s + s \cdot s = 0$

- norm cone: $\{ (x, s) \mid \|x\| \leq s \}$ ← not necessarily $\|\cdot\|_2$

  dual $= \{ (y, t) \mid \|y\|_* \leq t \}$

dual norm

$\|y\|_* = \max_{\{x \mid \|x\| \le 1\}} x \cdot y$

definition — e.g.

$\|x\| = \max_i |x_i|$

$\|y\|_* = \max_x x \cdot y \quad |-1 \le x_i \le 1|$

$= \sum_i |y_i|$

S_+ is self-dual

- S_+: { A | A=A^T, x^{T}Ax ≥ 0 for all x }

$S_+^* \subseteq S_+$ Suppose $Y \not\succeq 0 \Rightarrow u^T Y u < 0$

$\Rightarrow tr(u^T Y u) < 0 \Rightarrow tr(\underline{uu^T}\,\underline{Y}) < 0$

$\Rightarrow Y \notin S_+^*$

$S_+^* \supseteq S_+$ ~~pros~~ given $X \succeq 0$ show $tr(X^T Y) \ge 0$

for all $Y \succeq 0$

$X = \sum_i \lambda_i v_i v_i^T \quad \lambda_i \ge 0$

$tr(Y^T X) = \sum_i \lambda_i \, tr(Y^T v_i v_i^T) = \sum_i \underline{\lambda_i} \, \underline{tr(v_i^T Y^T v_i)} \ge 0$

Ex: Euclidean distance matrices

m

- Given points $x_i \in \mathbb{R}^n$
- Matrix D: $D_{ij} = \|x_i - x_j\|^2$

$K = \{ D \mid \quad \}$

$\in \mathbb{R}^{m \times m}$

$-K^* = \{ P + Q \mid P \succeq 0 \quad P = P^T \quad \sum_{ij} P_{ij} = 0 \quad Q \text{ diagonal} \}$

show $\mathrm{tr}(D^T \cdot (P+Q)) \geq 0$ $\qquad$ show $\mathrm{tr}(D^T (uu^T + Q)) \geq 0$

$1^T u = 0$

$\mathrm{tr}(D^T Q) = 0$ since $D_{ii} = 0 \ \forall i$

$\mathrm{tr}(D^T uu^T) = \mathrm{tr}(u^T D u) = \sum_{ij} u_i u_j (x_i^T x_i - 2 x_i^T x_j + x_j^T x_j)$

$= 2 \sum_{ij} u_i u_j x_i^T x_i - 2 \sum_{ij} u_i u_j x_i^T x_j$

$= 2 \sum_i u_i x_i^T x_i \underbrace{\sum_j u_j}_{=0} - 2 \left(\sum_i u_i x_i\right)^T \left(\sum_j u_j x_j\right) \leq 0$

Properties of dual cones

- K* is closed and convex

given y_1 y_2 s.t. $y_1^T x \geq 0$ $y_2^T x \geq 0$ $\forall x \in K$

$\Rightarrow (\alpha y_1 + (1-\alpha) y_2)^T x \geq 0 \Rightarrow \alpha y_1 + (1-\alpha) y_2 \in K^*$

given $y_1, y_2, \ldots$ $y_t^T x \geq 0$ $(\lim_{t\to\infty} y_t) \cdot x = \lim_{t\to\infty} y_t \cdot x \geq 0$

- K** = cl conv K

$K^{**} \supseteq$ cl conv K $\forall x \in K$ $x^T y \geq 0$ $\forall y \in K^*$ $\Rightarrow x \in K^{**}$

by closedness & convexity, $\lim x_t \in K^{**}$

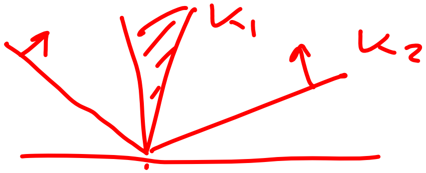
$\alpha x_1 + (1-\alpha) x_2 \in K^{**}$

$K^{**} \subseteq$ cl conv K : take $v \notin$ cl conv K $\Rightarrow \exists y.$ $\underbrace{y^T v < 0}_{v \notin K^{**}}$ $y^T x \geq 0$ $\underbrace{\forall x \in \text{cl conv } K}_{y \in K^*}$

- If K closed and convex, $K^{**} = K$

Properties of dual cones

- $K_1 \subseteq K_2 \Rightarrow K_2^* \subseteq K_1^*$



- $K_1 \supseteq K_2 \qquad K_2^* \supseteq K_1^*$
- If K_1, K_2 are closed and convex:

$$K_1 \subseteq K_2 \iff K_1^* \supseteq K_2$$

$$K_1 \subset K_2 \iff K_1^* \supset K_2^*$$