Solving convex programs

- Linear programs: \textit{Simplex, subgradient, ellipsoid, interior point}

- General CP: \textit{Simplex, subgradient, ellipsoid, interior point}

- Interesting special cases: QP, SOCP, SDP
Separation oracle: QPs

- \( \min_{x} q(x) \) s.t. \( Ax = b, \ x \geq 0 \)

\[
\begin{align*}
\min_{z, x} & \quad z \\
\text{s.t.} & \quad Ax = b, \ x \geq 0, \ z \geq q(x)
\end{align*}
\]

\[x_0, z_0 \quad \text{s.t.} \quad z_0 < q(x_0)\]
\[z_0 < q(x_0) + (x - x_0) \cdot q'(x_0)\]
\[z \geq q(x_0) + (x - x_0) \cdot q'(x_0) \quad \text{is violated by } x_0\]
\[z \geq q(x_0) + (x - x_0) \cdot q'(x_0) \quad \text{is satisfied by all feasible } x, z\]
Separation oracle: SOCPs

• SOC constraint: \(||Ax + b|| \leq c'x + d|

• Given \(x_0\) that fails:

\[
\begin{align*}
    u &= \frac{Ax_0 + b}{||Ax_0 + b||} \quad \text{or} \quad u = e_1 \quad \text{if} \quad Ax_0 + b = 0
\end{align*}
\]

new constr: \(u^T (Ax + b) \leq c^T x + d\)

plug \(x_0\):
\[
\begin{align*}
    u^T (Ax_0 + b) \leq c^T x_0 + d \quad \Rightarrow \quad ||Ax_0 + b|| \leq c^T x_0 + d \\
    \Rightarrow \quad \text{violated}
\end{align*}
\]

for any feasible \(x\):
\[
\begin{align*}
    u^T (Ax + b) \leq ||Ax + b|| \leq c^T x + 1
\end{align*}
\]
Separation oracle: SDPs

- SDP constraint:
  \[ A = x_1 A_1 + x_2 A_2 + \ldots \]
  \[ A \in S_+ \]

Given \( x_0, A_0 \), \( A_0 \neq 0 \) but \( A = \sum x_i A_i \)

\[ \exists u \neq 0, u^T A_0 u < 0 \implies A_0 \text{ violates } u^T A u \geq 0 \]
Convex duality

• Several new types of duality
  – convex cones
  – convex sets
  – convex functions
  – convex programs

• Generalize LP/QP duality

• Generalize norm duality (e.g., $L_1$ v. $L_\infty$)
Cone duality

• Cone \( K \) (not necessarily convex)
  \[ x \in K \Rightarrow \lambda x \in K \quad \lambda \geq 0 \]

• \( K^* = \{ y \mid x \cdot y \geq 0 \ \forall x \in K \} \)

Some books define \( K^* = \{ y \mid x \cdot y \leq 0 \ \forall x \in K \} \)

\( K^* \) = dual cone, polar cone

\( -y \) normal of halfspace containing \( K \)
\[ x \cdot (-y) \leq 0 \]
Examples of dual cones

• Halfspace $a^T x \geq 0$

• Subspace $\{ x \mid Ax = 0 \}$

• $\mathbb{R}^n_+$

• SOC: $\{ (x, s) \mid \|x\|_2 \leq s \}$

• norm cone: $\{ (x, s) \mid \|x\| \leq s \}$

\[ \{ 0 \} \Rightarrow \mathbb{R}^n \]

\[ \{ 2a \mid a \geq 0 \} \]

\[ \{ A^T y \mid y \in \mathbb{R}^n \} \]

\[ \mathbb{R}^n \]

\[ \mathbb{R}^n_+ \]

\[ x \geq 0, \ y \geq 0 \Rightarrow x^T y \geq 0 \]

\[ x = (3, -1, 2) \quad y = (0, 1, 0) \quad x^T y = -1 \]

\[ \text{soc}^* = \text{soc} \]

\[ (\chi^T, \chi^T) = x \cdot y + st \geq -\|x\|_2 \|y\|_x + st \geq 0 \]

\[ \text{if } x = y, \ s = t = \|x\| \Rightarrow -s \cdot s + s \cdot s = 0 \]

\[ \text{not necessarily } \|x\|_2 \]

\[ \text{dual} = \{ (y, t) \mid \|y\|_x \leq t \} \]
$S_+$ is self-dual

\[ S_+^* \subseteq S_+ \]

Suppose \( Y \neq 0 \) \( \Rightarrow u^TYu < 0 \)

\[ \Rightarrow \text{tr} \left( u^TYu \right) < 0 \]

\[ \Rightarrow \text{tr} \left( uu^TY \right) < 0 \]

\[ \Rightarrow y \notin S_+^* \]

\[ S_+^* \supseteq S_+ \]

Given \( X \geq 0 \) show \( \text{tr}(X^TY) \geq 0 \) for all \( Y \geq 0 \)

\[ X = \sum \lambda_i v_i v_i^T \quad \lambda_i \geq 0 \]

\[ \text{tr} \left( Y^TX \right) = \sum \lambda_i \text{tr} \left( Y^T v_i v_i^T \right) = \sum \lambda_i \text{tr} \left( v_i^T Y v_i \right) \geq 0 \]
Ex: Euclidean distance matrices

Given points $x_i \in \mathbb{R}^n$

Matrix $D$: $D_{ij} = \|x_i - x_j\|^2$

$\mathcal{V} = \{D | ? \}$

$\mathcal{V}^* = \{ P + Q \mid P \succeq 0 \ P = P^T \ \xi_i \text{i.e. } P_{ii} = 0 \ Q \text{ diagonal} \}$

Show $\text{tr}(D^T (P + Q)) \geq 0$

Show $\text{tr}(D^T (uu^T + Q)) \geq 0$

$1^T u = 0$

$\text{tr}(D^T Q) = 0$ since $D_{ii} = 0 \ \forall i$

$\text{tr}(D^T uu^T) = \text{tr}(u^T D u) = \sum_{ij} u_i u_j (x_i^T x_i - 2 x_i^T x_j + x_j^T x_j)$

$= 2 \sum_{ij} u_i u_j x_i^T x_i - 2 \sum_{ij} u_i u_j x_i^T x_j$

$= 2 \sum_{i} u_i x_i^T x_i \sum_{j} u_j - 2 \left( \sum_{i} u_i x_i \right)^T \left( \sum_{j} u_j x_j \right) \leq 0$

$= 0$
Properties of dual cones

• $K^*$ is closed and convex

\[
\begin{align*}
\text{minimize } & \quad y_1, y_2 \text{ s.t. } y_1^T x \geq 0, \quad y_2^T x \geq 0 \quad \forall x \in K \\
\Rightarrow & \quad (\alpha y_1 + (1-\alpha)y_2)^T x \geq 0 \quad \Rightarrow \quad \alpha y_1 + (1-\alpha)y_2 \in K^* \\
\text{given } y_1, y_2, \ldots & \quad y_T^T x \geq 0 \quad \lim_{t \to \infty} y_t \cdot x \geq 0 \\
\end{align*}
\]

• $K^{**} = \text{cl conv } K$

\[
\begin{align*}
K^{**} \supseteq \text{cl conv } K \quad \forall x \in K \quad x^T y \geq 0 \quad \forall y \in K^{**} \quad \Rightarrow \quad x \in K^{**} \\
\text{by closedness + convexity, } \lim_{t \to \infty} x_t \in K^{**} \\
K^{**} \subseteq \text{cl conv } K : \text{ take } v \notin \text{cl conv } K \Rightarrow \exists y. \quad y^T v < 0 \quad y^T x \geq 0 \quad \forall x \in K^{**} \\
\end{align*}
\]

• If $K$ closed and convex,

\[
K^{**} = K
\]
Properties of dual cones

- \( K_1 \subseteq K_2 \Rightarrow K_2^* \subseteq K_1^* \)

- \( K_1 \supseteq K_2 \quad K_2^* \supseteq K_1^* \)

- If \( K_1, K_2 \) are closed and convex:

\[
\begin{align*}
K_1 \subset K_2 & \iff K_1^* \supseteq K_2^* \\ 
K_1 \subsetneq K_2 & \iff K_1^* > K_2^*
\end{align*}
\]