Slater's condition: proof

- $p^* = \inf_x f(x)$ s.t. $Ax = b$, $g(x) \le 0$

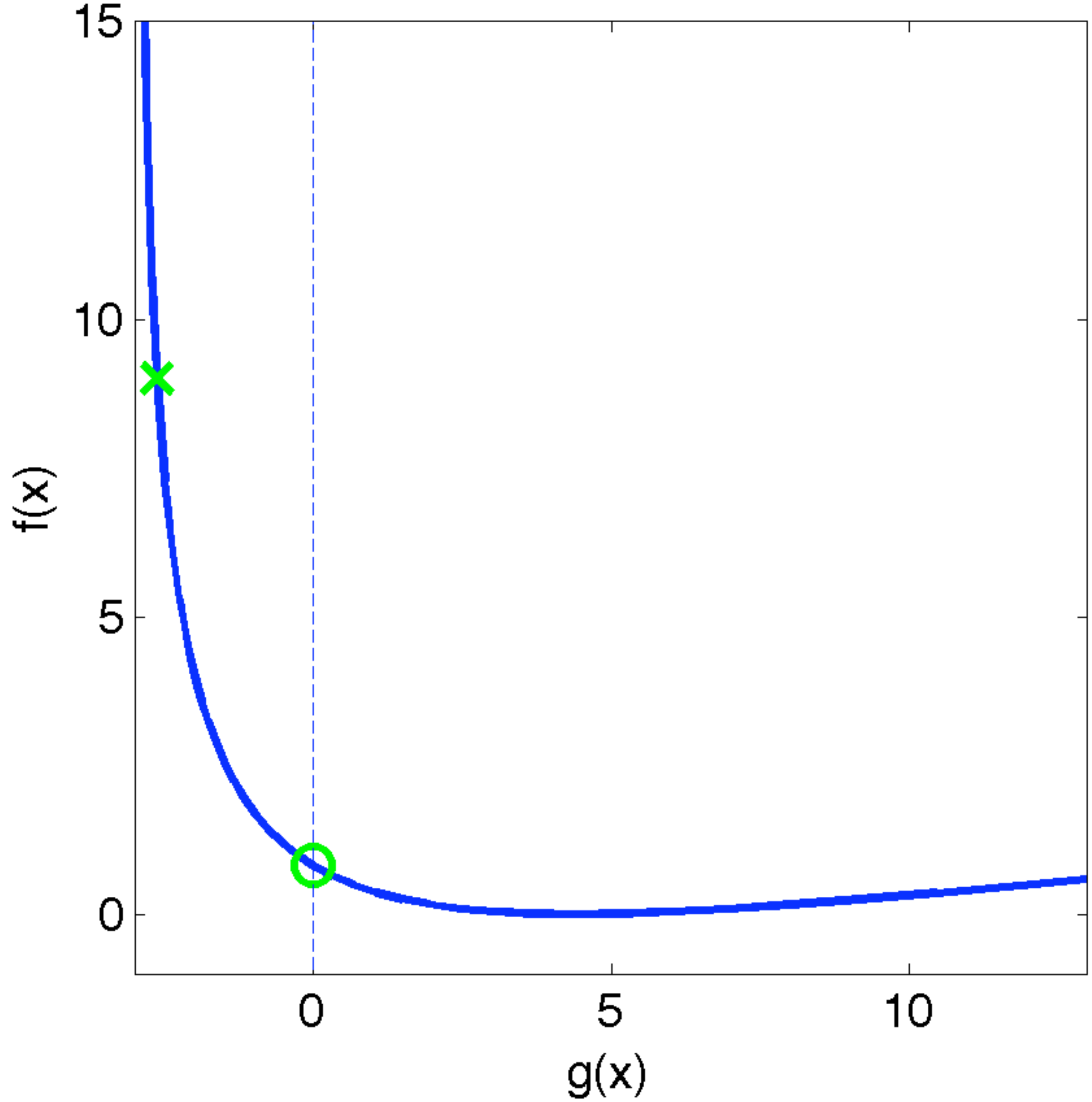
  e.g., $\inf x^2$ s.t. $e^{x+2} - 3 \le 0$

- A =

  e.g., A =

Picture of set A

L(y,z) =



Nonconvex example

Interpretations

$$L(x, y, z) = f(x) + y^T(Ax - b) + z^T g(x)$$

- Prices or sensitivity analysis

- Certificate of optimality

Optimality conditions

- $L(x, y, z) = f(x) + y^T(Ax–b) + z^Tg(x)$
- Suppose strong duality, (x, y, z) optimal

Optimality conditions

- $L(x, y, z) = f(x) + y^T(Ax-b) + z^Tg(x)$
- Suppose (x, y, z) satisfy KKT:

$$Ax = b \qquad g(x) \leq 0$$

$$z \geq 0 \qquad z^Tg(x) = 0$$

$$0 \in \partial f(x) + A^Ty + \sum_i z_i \, \partial g_i(x)$$

Using KKT

- Can often use KKT to go from primal to dual optimum (or vice versa)
- E.g., in SVM:

  $\alpha_i > 0 \Longleftrightarrow y_i(x_i^T w + b) = 1$
- Means $b = y_i - x_i^T w$ for any such i
  - typically, average a few in case of roundoff

Set duality

- Let C be a set with $0 \in \text{conv}(C)$
- $C^* = \{ y \mid x^T y \leq 1 \text{ for all } x \in C \}$
- Let $K = \{ (x, s) \mid x \in sC \}$
- $K^* = \{ (y, t) \mid x^T y + st \geq 0 \text{ for all } X \in K \}$

What is set duality good for?

- Related to norm duality
- Useful for helping visualize cones

Duality of norms

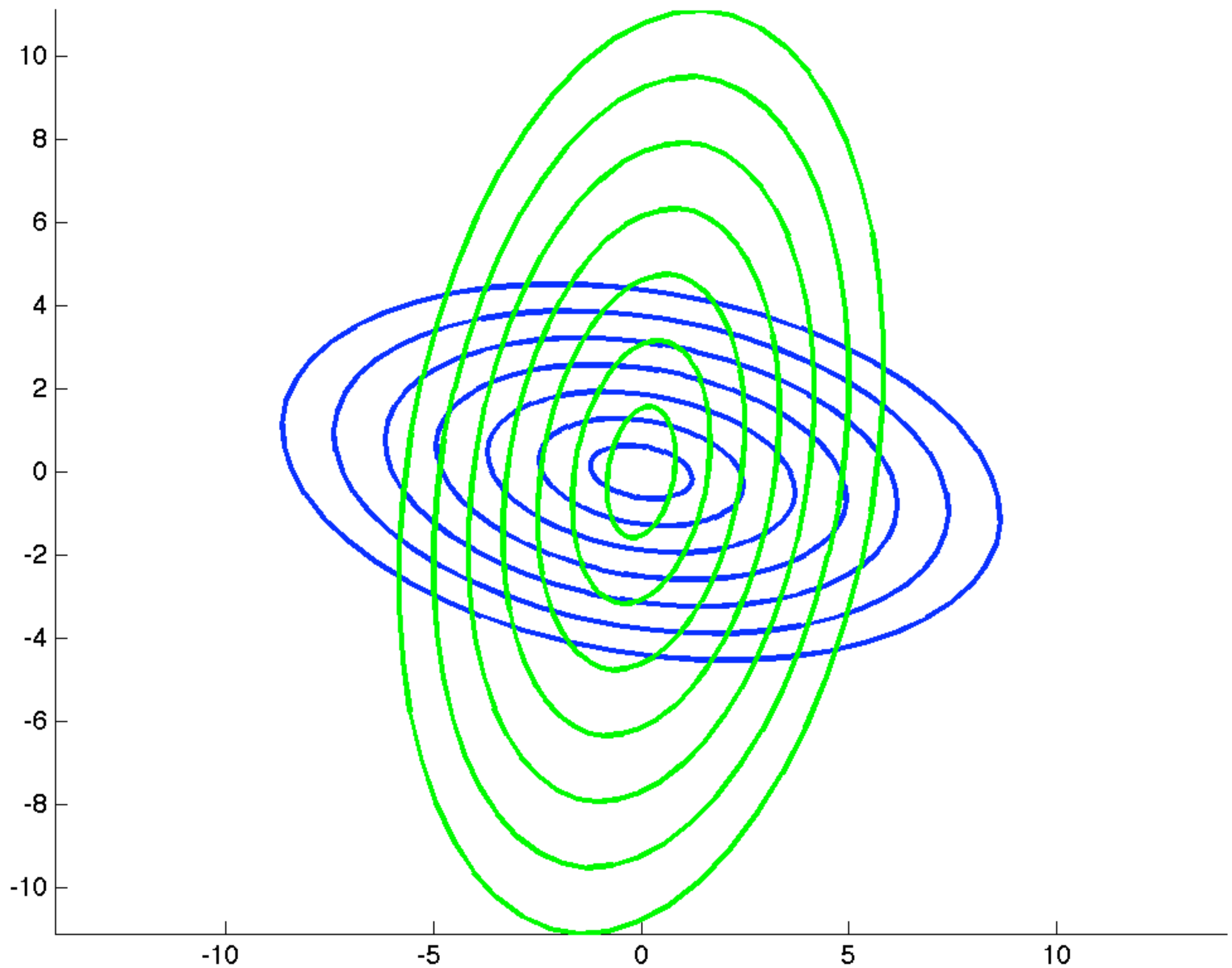
- Dual norm definition

  $||y||_* = \max$

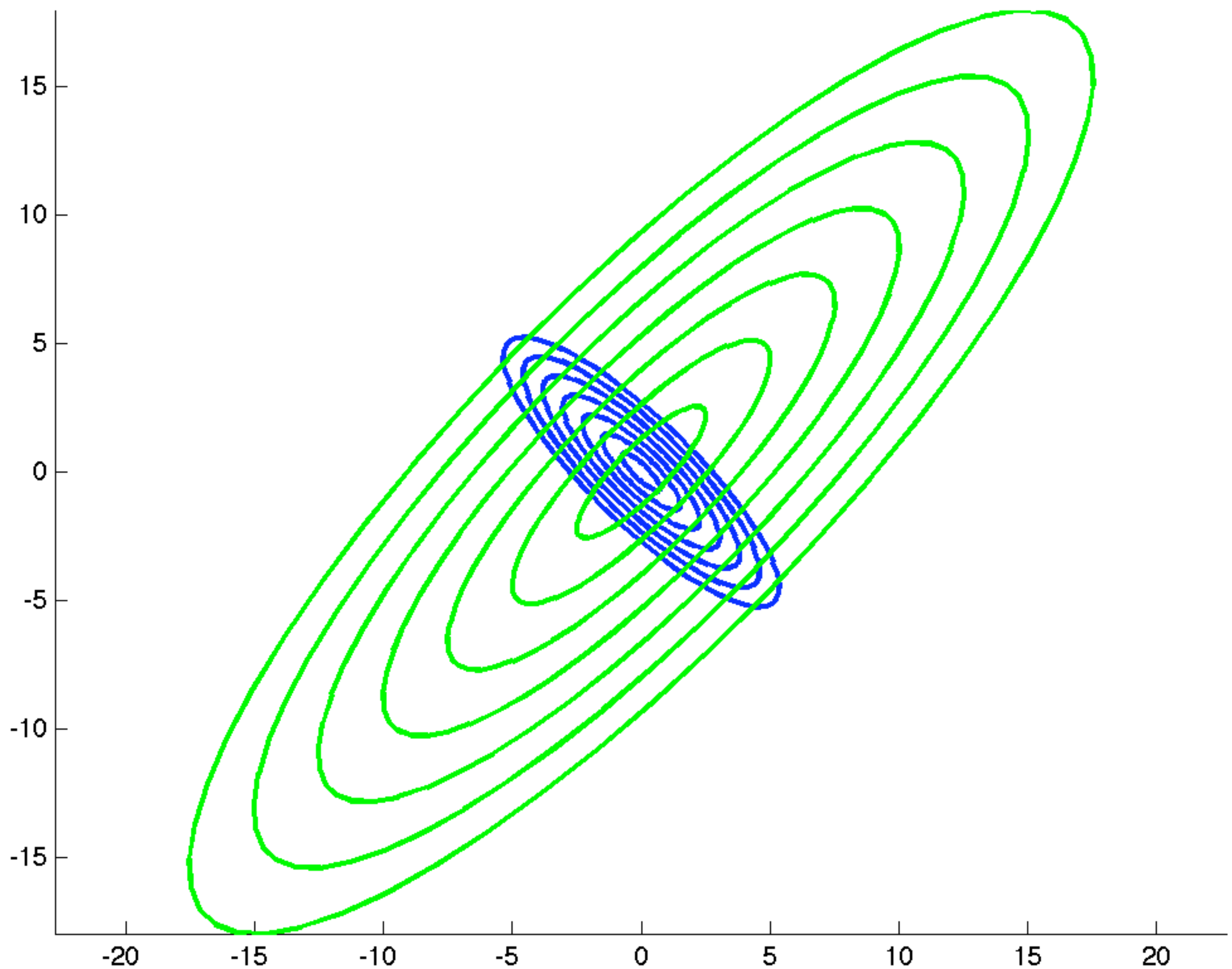
- Motivation: Holder's inequality

  $x^T y \leq ||x||\ ||y||_*$

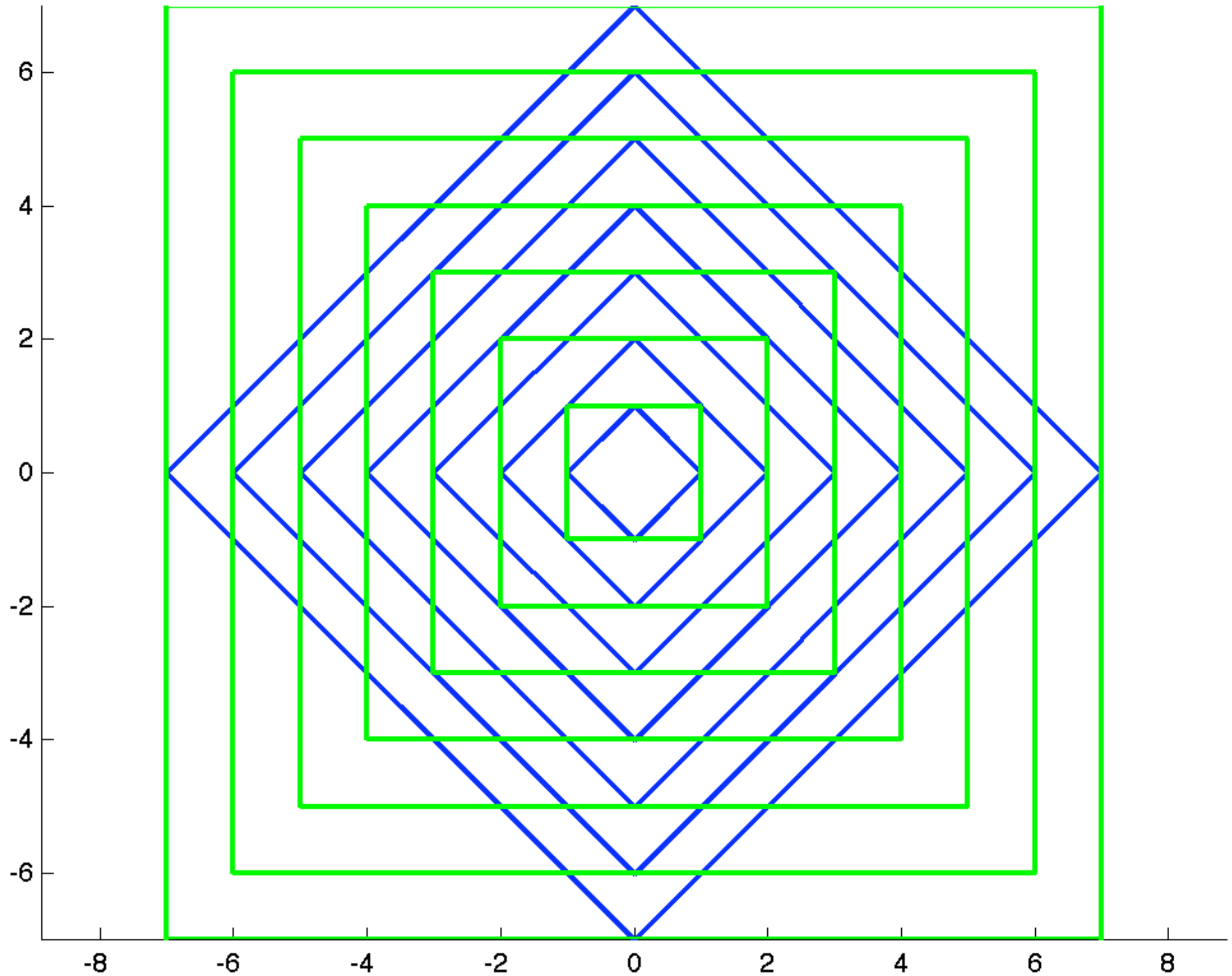
Dual norm examples



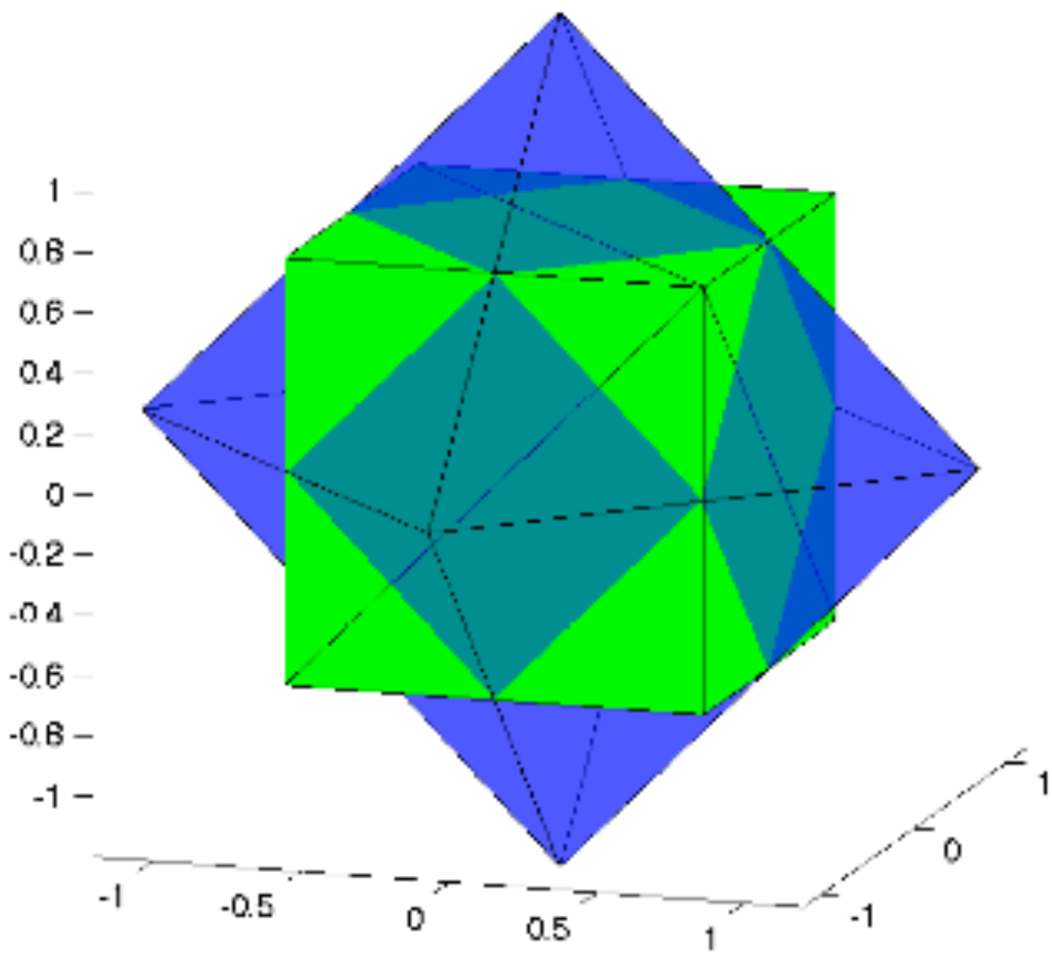
Dual norm examples



Dual norm examples



Cuboctahedron



$||y||_*$ is a norm

- $||y||_* \geq 0$:

- $||ky||_* = |k|\ ||y||_*$:

- $||y||_* = 0$ iff $y = 0$:

- $||y_1+y_2||_* \leq ||y_1||_* + ||y_2||_*$

Dual-norm balls

- $\{ y \mid ||y||_* \leq 1 \} =$

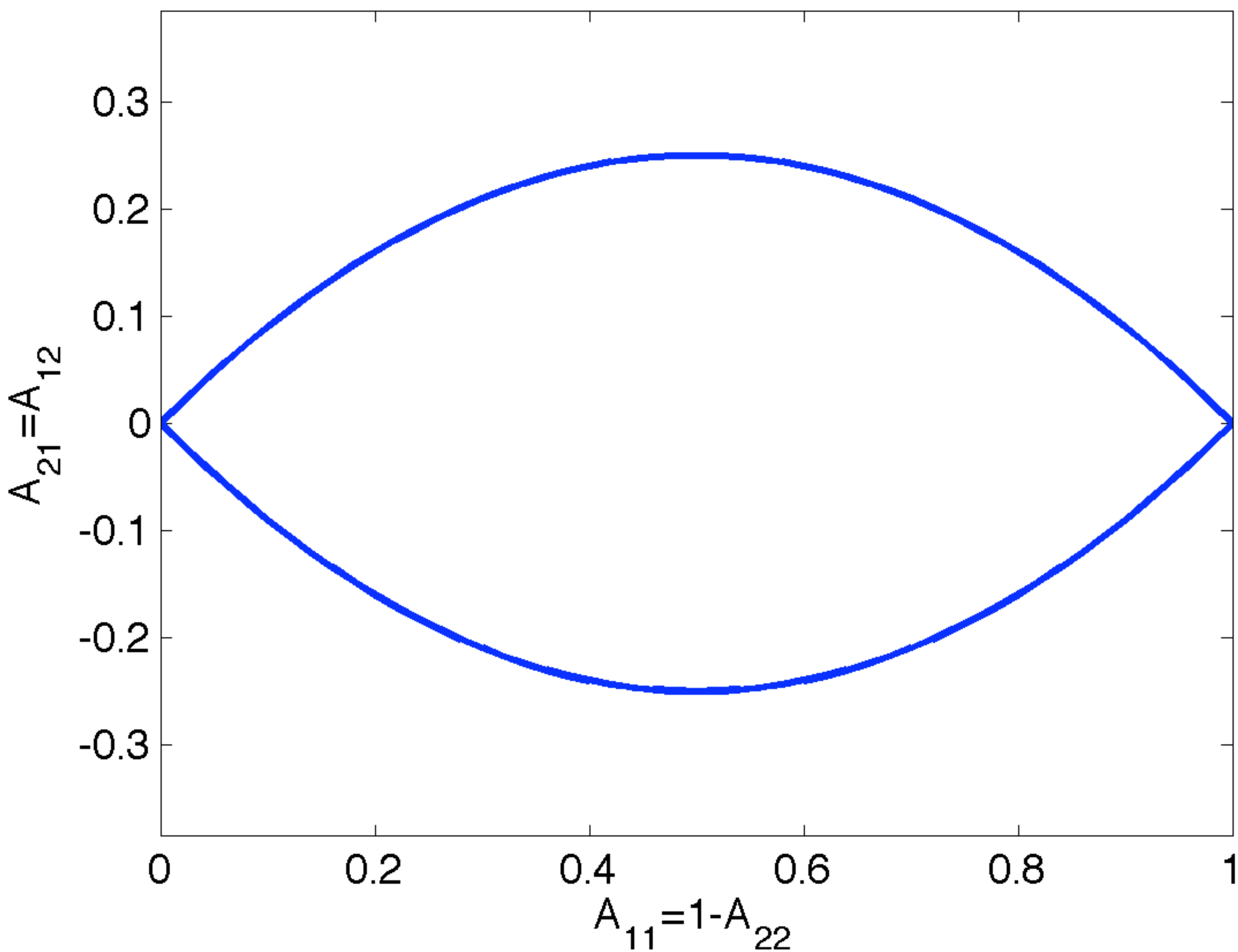
- Duality of norms:

Visualizing cones

- Suppose we have some weird cone in high dimensions (say, $K = S_+$)
- Often easy to get a vector u in $K \cap K^*$
  - e.g., $I \in S_+$, $I \in S_+^* = S_+$
- Plot $K \cap \{ u^T x = 1 \}$ and $K^* \cap \{ u^T x = 1 \}$ instead of K, K*
  - saves a dimension

Visualizing S_+

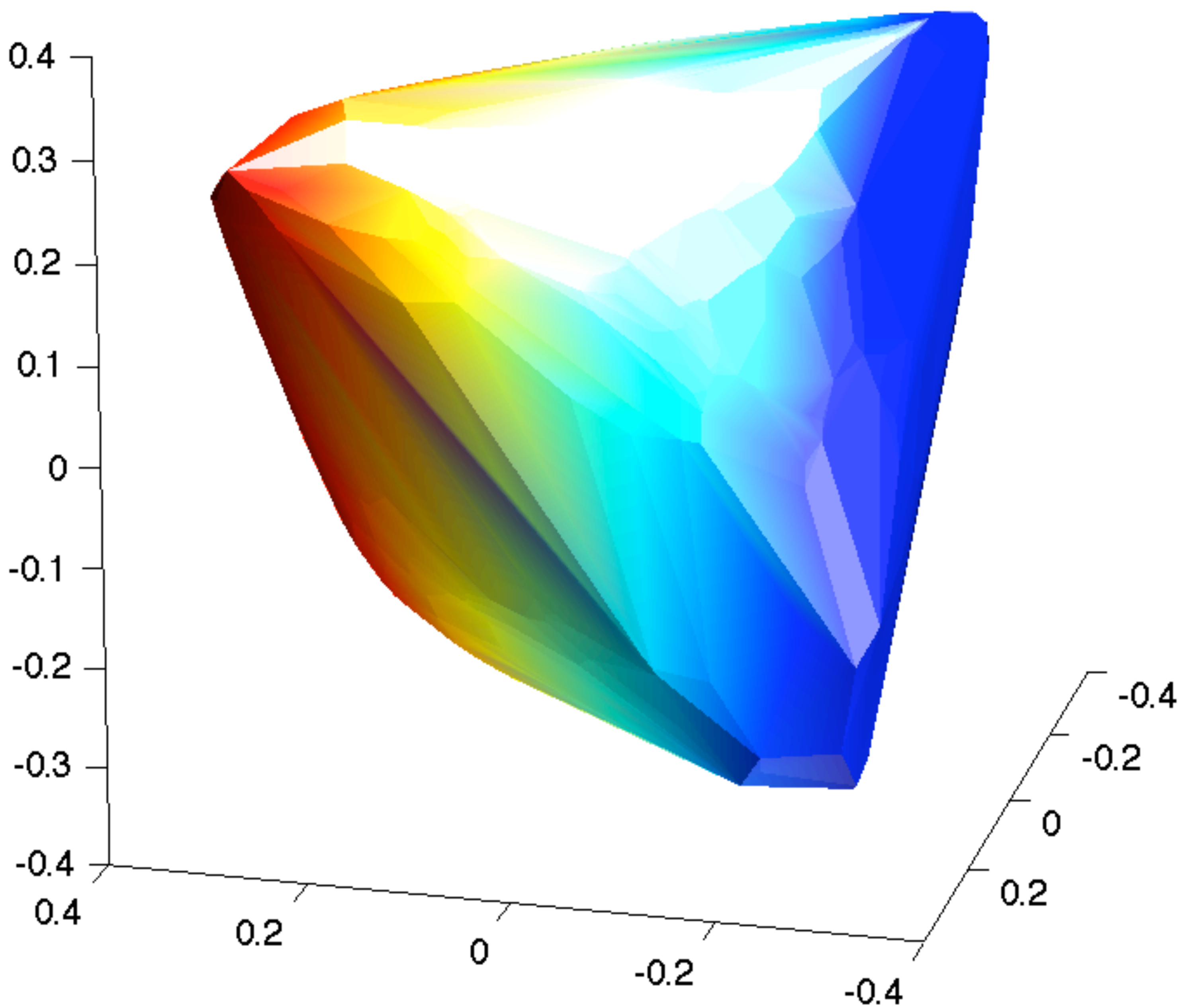
- Say, 2 x 2 symmetric matrices
- Add constraint $tr(X^T I) = 1$
- Result: a 2D set



0.3
0.2
0.1
0
-0.1
-0.2
-0.3
A21=A12
0
0.2
0.4
0.6
0.8
1
A11=1-A22

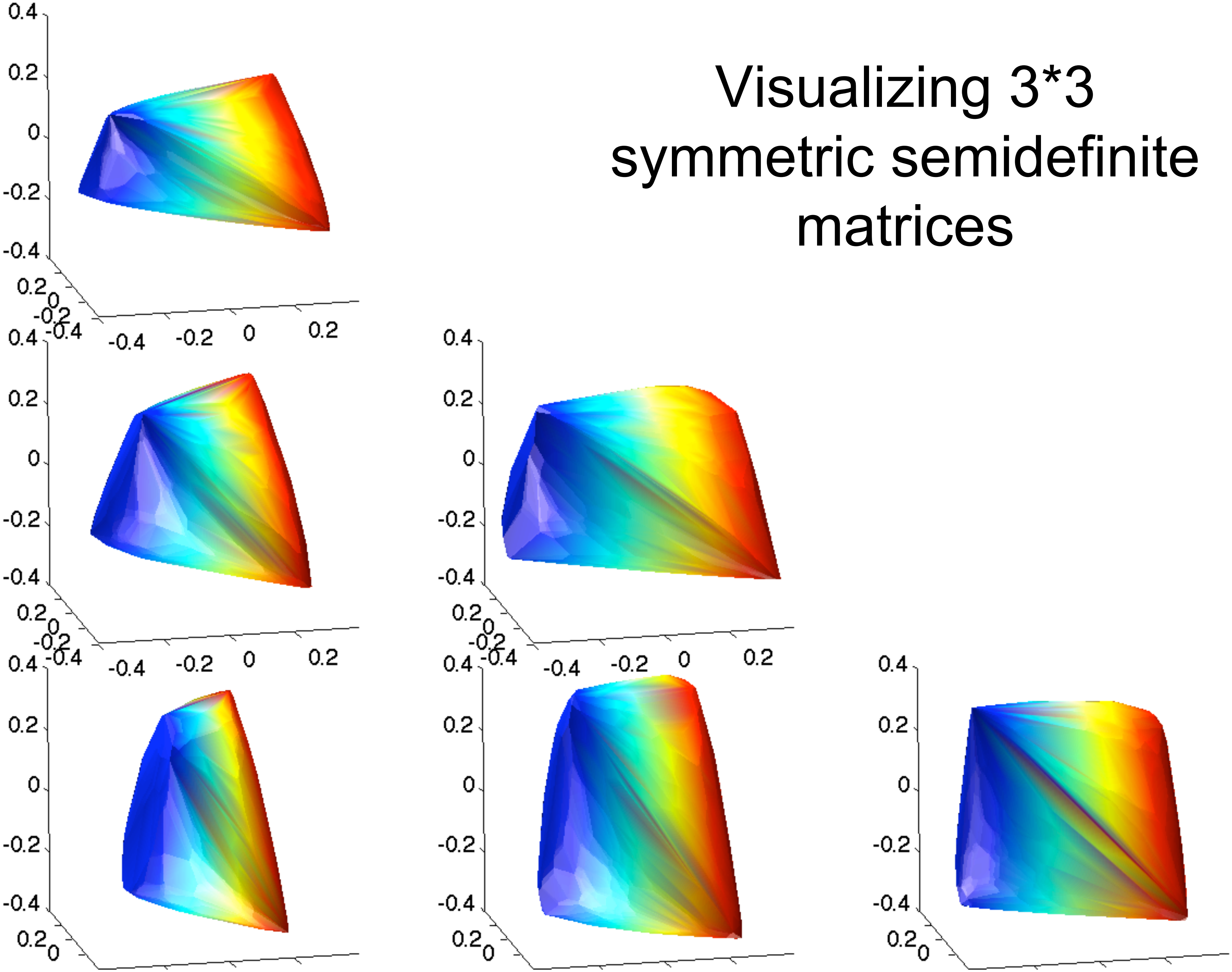
What about 3 x 3?

- 6 parameters in raw form
- Still 5 after tr(X)=1
- Try setting entire diagonal to 1/3
  - plot off-diagonal elements



0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
0.4
0.2
0
-0.2
-0.4
-0.4
-0.2
0
0.2

Visualizing 3*3 symmetric semidefinite matrices



Multi-criterion optimization

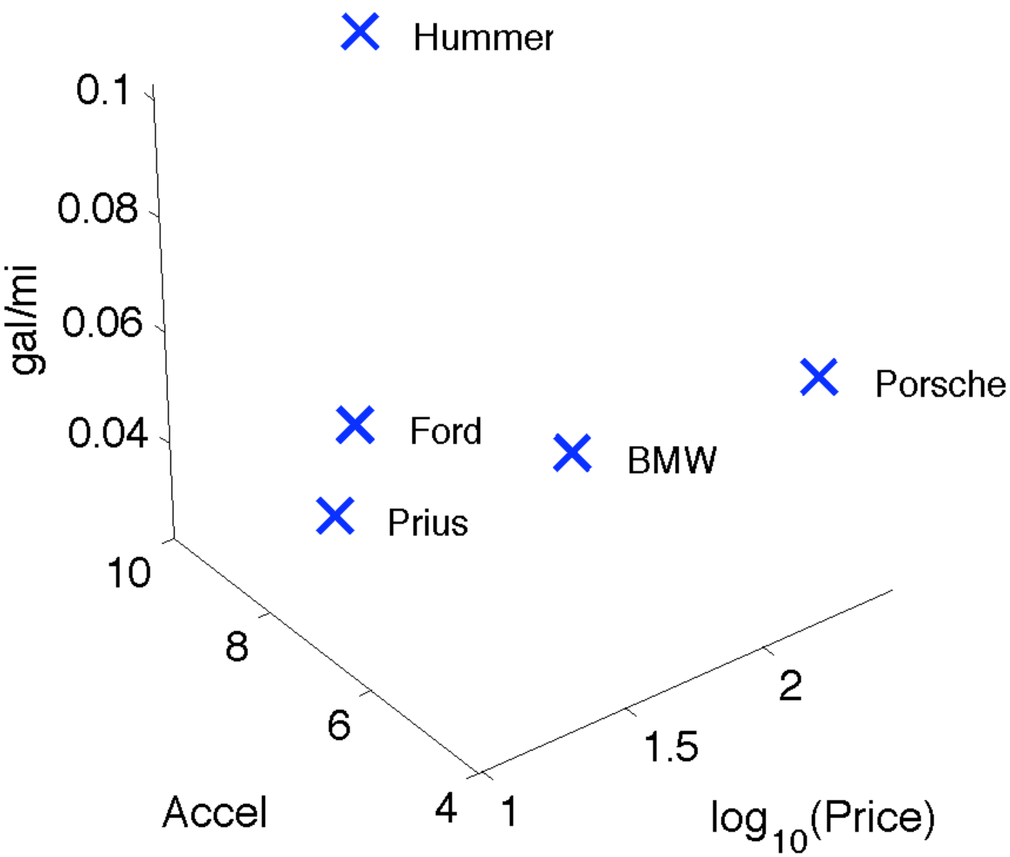
- Ordinary feasible region
- Indecisive optimizer: wants all of

Buying the perfect car

$K 0-60 MPG

Pareto optimality

x* Pareto optimal =



Pareto examples

Scalarization

- To find Pareto optima of convex problem: