$v^* = \text{opt}$

$f(x) = \infty$ for $x \notin \text{dom } f$

Convex program duality

f, g_i: convex

- min $f(x)$ s.t.

  $Ax = b$

  $g_i(x) \leq 0 \qquad i \in I$

$g(x) := \begin{pmatrix} g_1(x) \\ g_2(x) \\ \vdots \end{pmatrix}$

for feasible x

$y^T(Ax-b) = 0$

$z^T g(x) \leq 0$ if $z \geq 0$

$f(x) \geq f(x) + y^T(Ax-b) + z^T g(x)$

$\equiv L(x,y,z)$

$\rightarrow v^* = \inf_{\text{feas } x} f(x) \geq \inf_{\text{feas } x} L(x,y,z) \geq \inf_x L(x,y,z) \equiv L(y,z)$

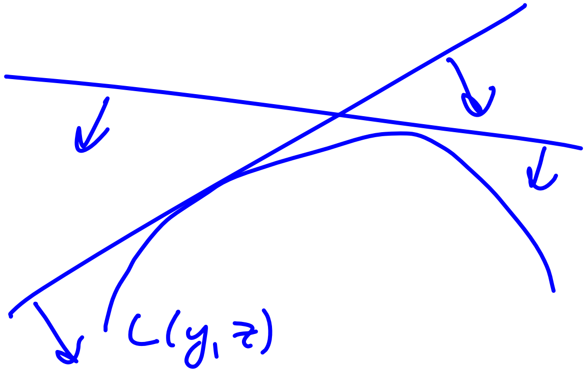
$v^* \geq d^*$

Dual prob: $\max L(y,z)$ s.t. $z \geq 0$

d^* = dual optimal value

Properties

- Weak duality $v^* \geq d^*$



- L(y, z) is closed, concave

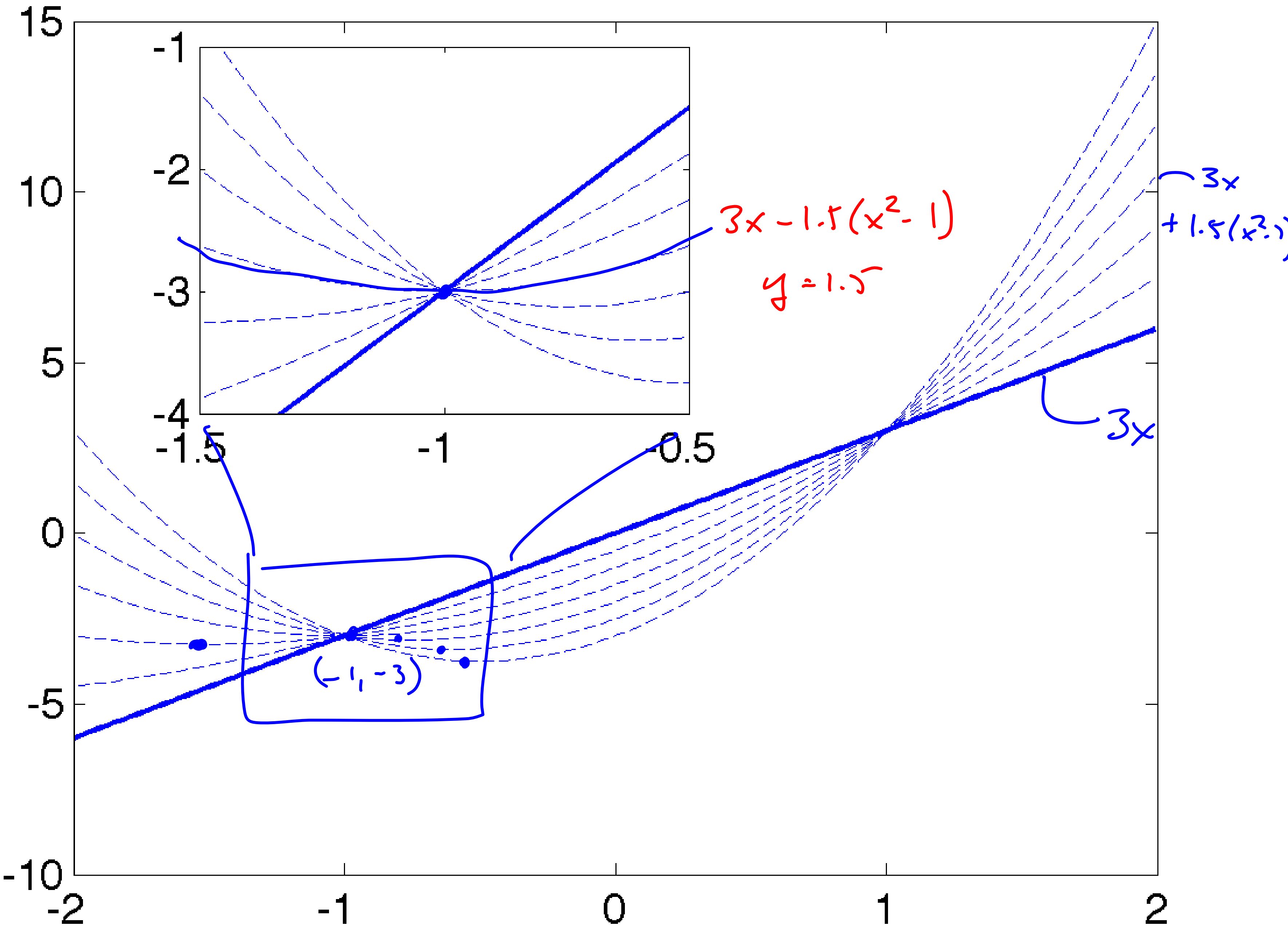
$$L(y,z) = \inf_x \left(f(x) + y^T(Ax-b) + z^T g(x) \right)$$

fixed x: linear fn of y, z

inf of linear fns: L(y, z) concave closed

Duality example

- min $\underline{3x}$ s.t. $\underline{x^2 \leq 1}$
- $L(\underbrace{x}_{\text{primal}}, \underbrace{y}_{\text{dual}}) = 3x + y(x^2 - 1)$ $\quad y \geq 0$



15
10
5
0
-5
-10
-2
-1
0
1
2
-1
-2
-3
-4
-1.5
-1
-0.5
3x - 1.5(x² - 1)
y = 1.5
3x
+ 1.5(x² - 1)
3x
(-1, -3)

Dual function

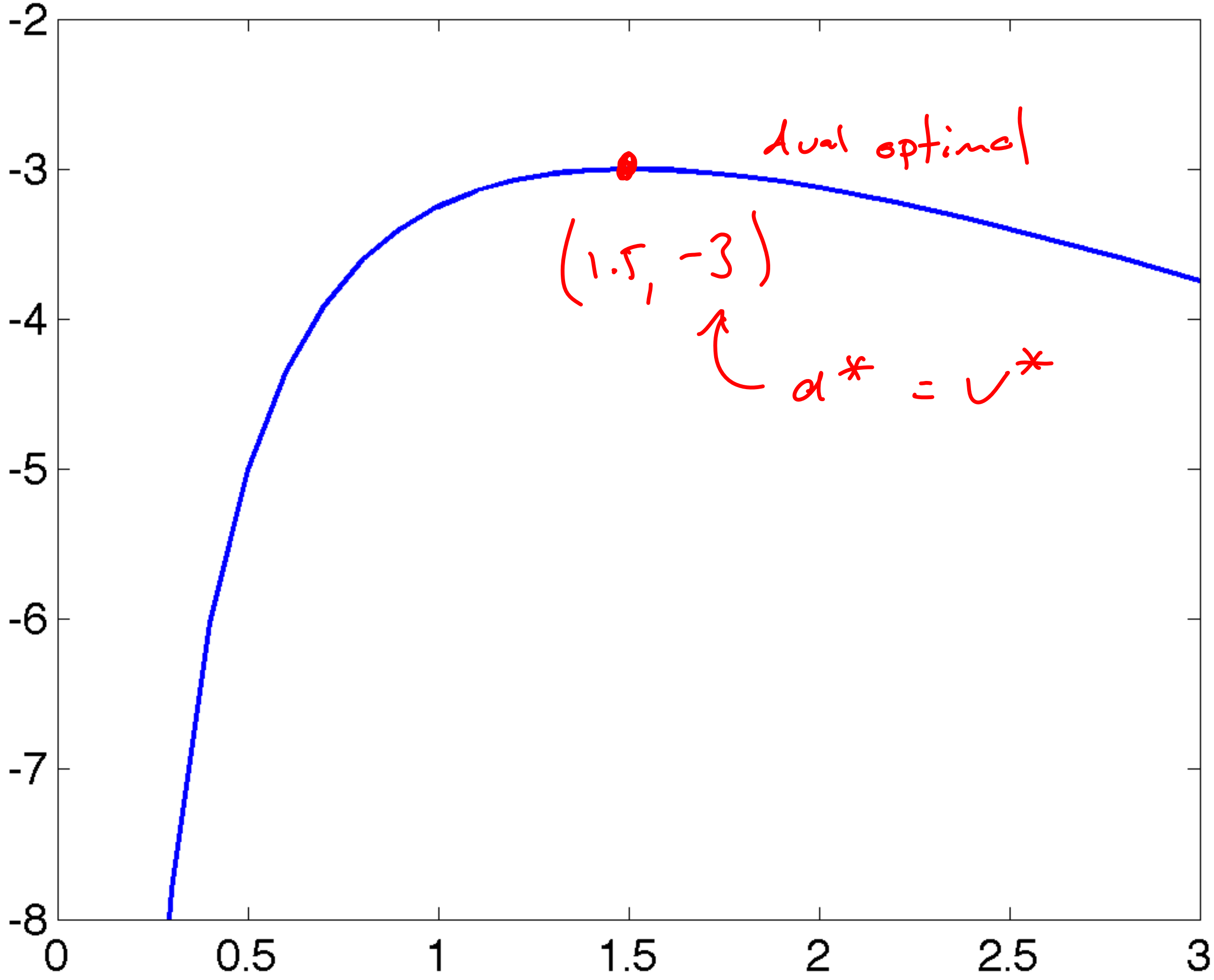
- $L(y) = \inf_x L(x,y) = \inf_x 3x + y(x^2 - 1)$

$$0 = \frac{d}{dx} = 3 + 2xy$$

$$x = \frac{-3}{2y}$$

$$L(y) = \underbrace{3 \cdot \left(\frac{-3}{2y}\right)}_{-\frac{9}{2y}} + \underbrace{y\left(\frac{9}{4y^2} - 1\right)}_{\frac{9}{4y} - y} = -\frac{9}{4y} - y$$

Dual function



Duality w/ linear constraints

convex f

- min f(x) s.t. Ax = b, Cx ≤ d
- L(x, y, z) = $f(x) + y^T(Ax - b) + z^T(Cx - d) \quad z \geq 0$
- L(y, z) = $\inf_x \left[f(x) + x^T(A^T y + C^T z) - b^T y - d^T z \right]$

$$= -\underbrace{\sup_x \left[-f(x) + x^T(-A^T y - C^T z) \right]}_{f^*(-A^T y - C^T z)} - b^T y - d^T z$$

- Dual problem

$$\max \ -f^*(-A^T y - C^T z) - b^T y - d^T z \quad \text{s.t. } z \geq 0$$

CP duality w/ cone constraints

$g_i(x) = I_K(A_i x + b_i) \le 0$

- min $f(x)$ s.t.

$A_i x + b_i \geq_K 0$

$A_0 x + b_0 = 0$

$A_i x + b_i \in K \quad i \in I$

$\forall$ feas x. $y^T(A_0 x + b_0) = 0$

$z_i^T (A_i x + b_i) \ge 0$

if $z_i \in K^*$

$y^T(A_0 x + b_0) + \sum_i z_i^T (A_i x + b_i) \ge 0 \quad z_i \in K^*$

$x^T(A_0^T y + \sum A_i^T z_i) \ge -b_0^T y - \sum b_i^T z_i$

$f(x) \ge w^T x - f^*(w)$ take $w = A_0^T y + \sum A_i^T z_i$

$f(x) \ge -b_0^T y - \sum b_i^T z_i - f^*(w)$

- Dual: $\max_y \; -b_0^T y - \sum b_i^T z_i - f^*(A_0^T y + \sum A_i^T z_i)$

$z_i \in K^*$

Ex: 2nd order cone program

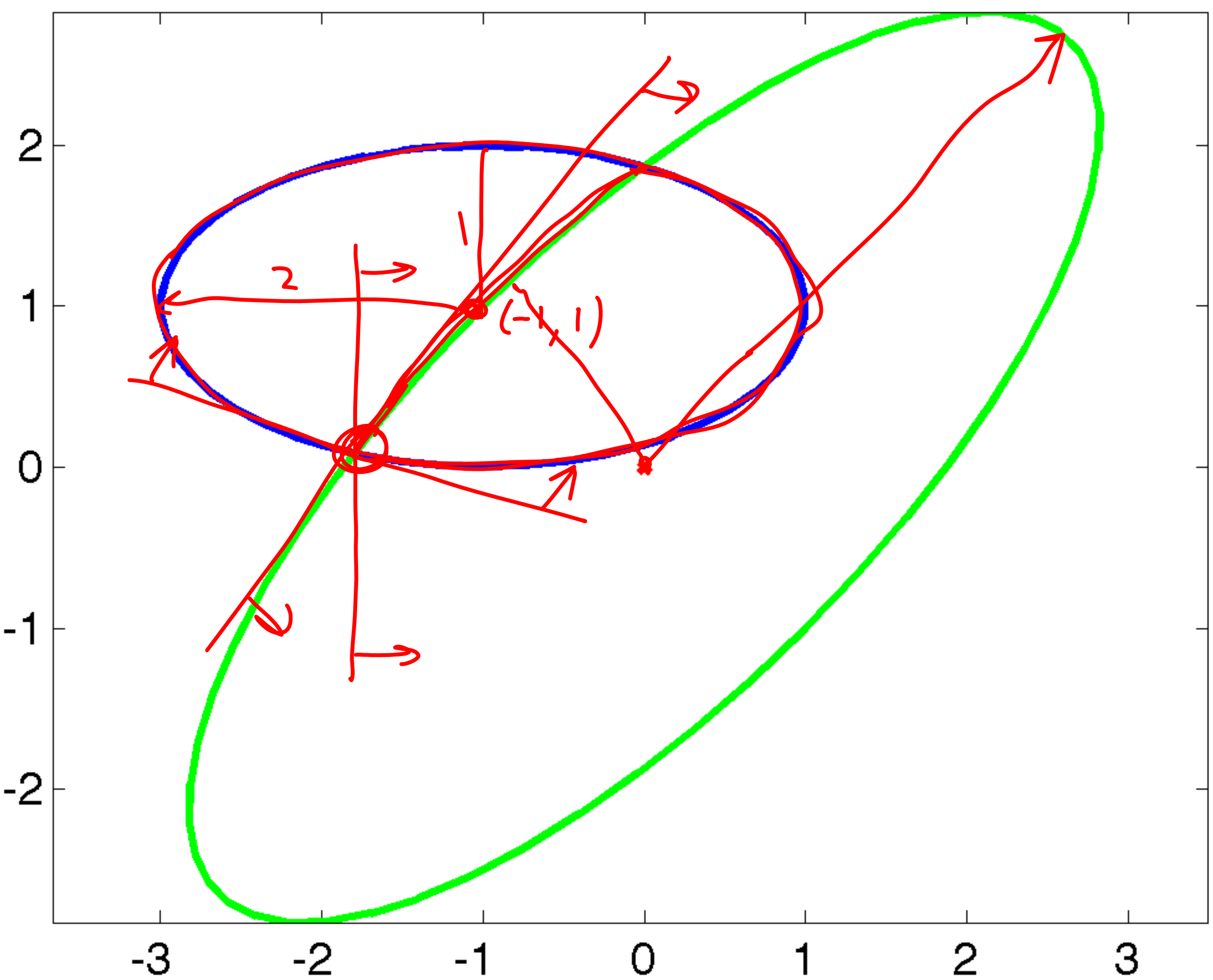
$K = SOC \quad \min x$

$$A_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$b_1 = \begin{pmatrix} +1/2 \\ -1 \\ 1 \end{pmatrix} \begin{matrix} \leftarrow x_c = -1 \\ \leftarrow y_c = 1 \\ \end{matrix}$$

$$A_2 = \begin{pmatrix} 1/2 & 1/2 \\ 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$b_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



Strong duality

- $v^* = \inf_x f(x)$ s.t. $Ax=b$, $g(x) \leq 0$

  $L(x, y, z) =$ $f(x) + y^T(Ax-b) + z^T g(x)$ $z \geq 0$

  $L(y, z) =$ $\inf_x L(x,y,z)$
- $d^* = \sup_{yz} L(y, z)$ s.t. $z \geq 0$
- Strong duality: $v^* = d^*$ (weak duality $v^* \geq d^*$)
- Slater's condition: $\Rightarrow$ strong duality
  - f, g_i: convex
  - strictly feasible point $\exists x \in \text{rel int dom } f$. $g(x) < 0$

Slater's condition, part II

- If $g_i(x)$ affine, only need $g_i(x) \le 0$

Can mix & match: $\le$ for all affine g_i

$<$ for all other g_i

- E.g., for $\inf_x f(x)$ s.t. $Ax = b$, $Cx \le d$

convex ~~nonempty~~

non-empty domain $\Rightarrow$ strong duality

$\exists x .\ Cx \le d \wedge f(x) < \infty \wedge Ax = b$

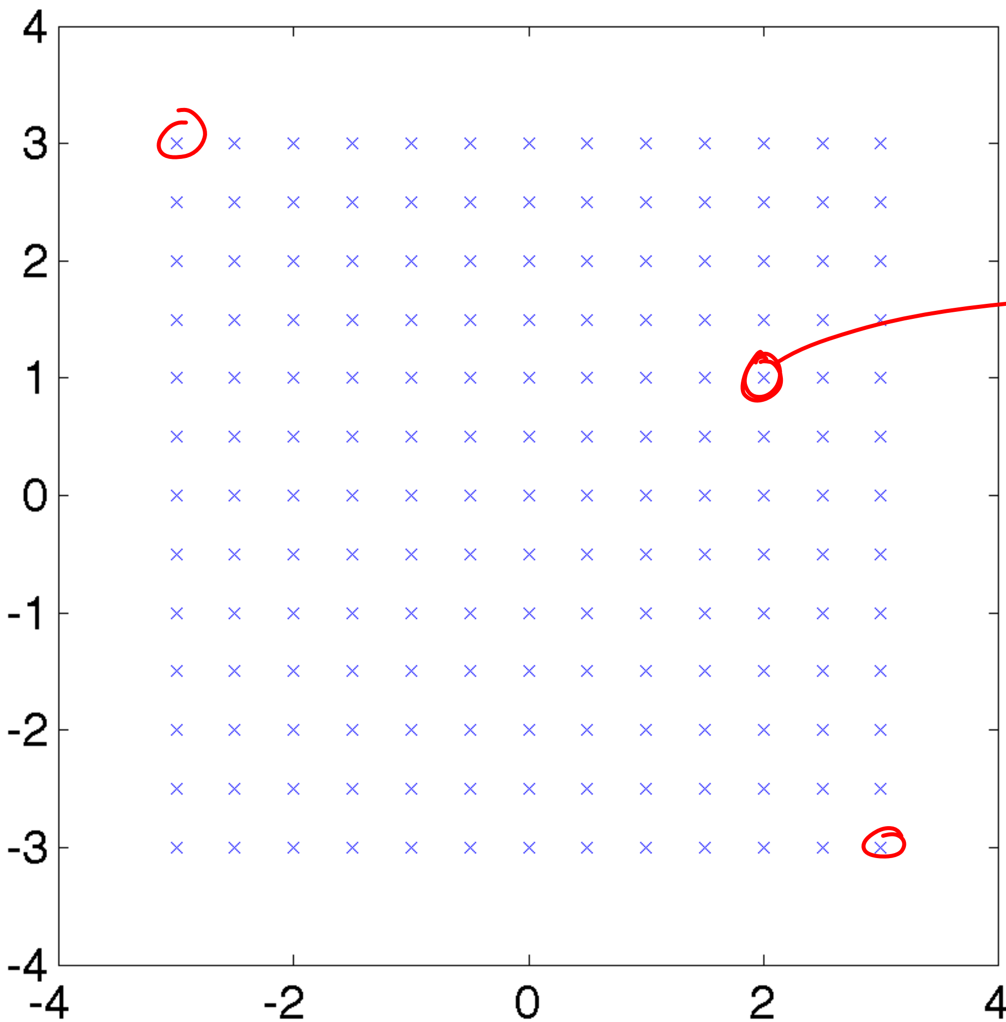
Slater's condition: sufficient but not necessary for strong duality

Example: maxent

$p \in \mathbb{R}^{169} \quad p \geq 0$

$\sum_i p_i = 1$

$T^T p = E_p(t) \quad E_p(t_i) = 1$

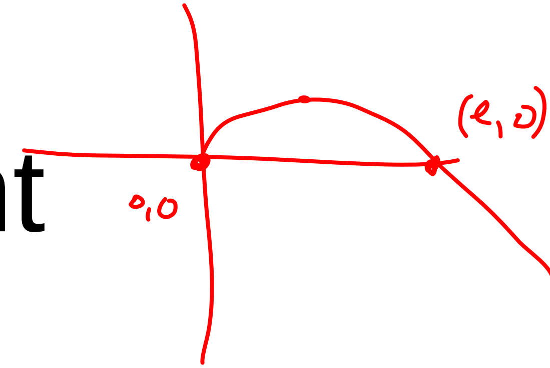


$$t_i = \begin{pmatrix} 1 \\ x \\ y \\ x^2 \\ y^2 \\ xy \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 4 \\ 1 \\ 2 \end{pmatrix}$$

6

$$T = \begin{array}{c} \\ 169 \end{array} \begin{pmatrix} 1 & -3 & +3 & 9 & 9 & 9 \\ & & \vdots & & & \\ 1 & 2 & 1 & 4 & 1 & 2 \\ & & \vdots & & & \\ 1 & 3 & -3 & 9 & 9 & 9 \end{pmatrix}$$

Example: maxent



- Maxent problem:

  max H(p) s.t. T'p = b $= T^{T} p_0$

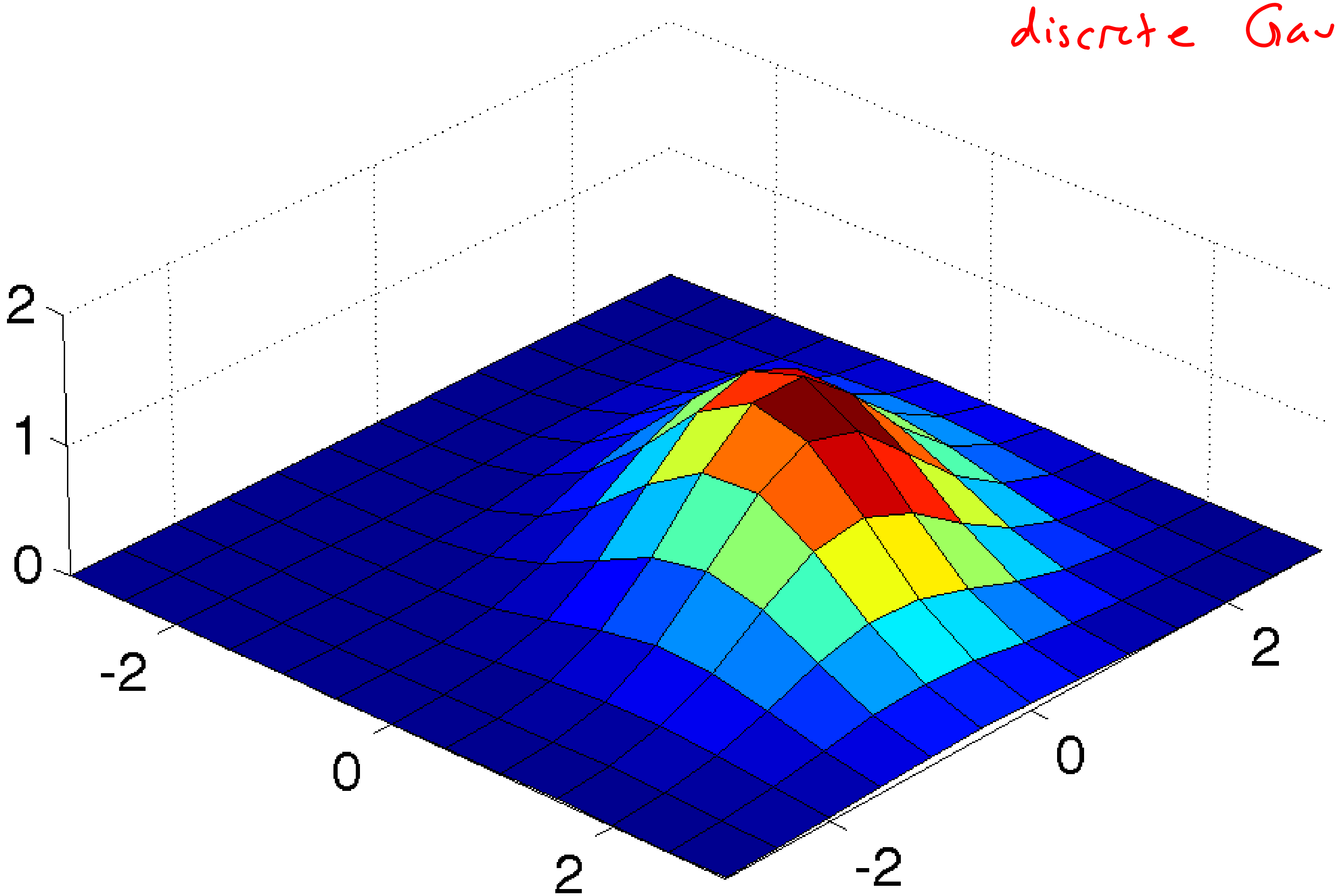
  H(p) = $\sum_i (p_i - p_i \ln p_i)$

  H'(p) = $\frac{dH}{dp_i} = \cancel{1} - \cancel{p_i \frac{1}{p_i}} - \ln p_i = -\ln p_i \quad H'(p) = -\ln p$

- Slater's condition: holds → ~~[illegible]~~ p_0

Example of maxent solution

discrete Gaussian



Dual of maxent

- max H(p) s.t. $T^T p = b$ $= T^T p_0$
- $L(p, y) =$ $H(p) + y^T(T^T p - b)$
- $L(y) = \cancel{\inf}\ \sup_p L(p, y) =$

$0 = \frac{d}{dp} L(p,y) = -\ln p + Ty \Rightarrow p = e^{Ty}$ exponential family

$L(y) = 1^T e^{Ty} - \cancel{(e^{Ty})^T Ty} + y^T(\cancel{T^T e^{Ty}} - b) = 1^T e^{Ty} - y^T b$

Dual: $\min \left(1^T e^{Ty} - y^T b\right) = L(y)$

$\frac{d}{dy_i} L(y) = 1^T e^{Ty} - b_i \overset{!}{=} 0 \quad 1^T e^{Ty} = 1$

Rev. dual. $\min -y^T b$ s.t. $1^T e^{Ty} = 1$ maximum likelihood in exponential family

$p = \exp(Ty)$ $\max y^T T^T p_0 = p_0^T \ln p$

$Ty = \ln(p)$

Is Slater necessary? No

$A = USU^T$ (S diag) $A = A^T$ $A \not\succeq 0$ $x \in \mathbb{R}^n$

- $\min_x \; x^TAx + 2b^Tx$ s.t. $||x||^2 \le 1$

trust region opt.

relax $\min_{x\,X} \; tr(X^TA) - 2b^Tx$ s.t. $X = X^T$ $tr(X) \le 1$ $\begin{pmatrix} X & x \\ x^T & 1 \end{pmatrix} \succeq 0$

if $X = xx^T$ $tr(X) \le 1 \iff x^Tx \le 1$ $1 \ge 0$

Slater: $X = I/2n$ $x = 0$ x^TAx $X - xx^T \succeq 0$

if $X = xx^T$

$tr(X^TA) = x^TAx$

Suppose (at opt) $X = xx^T + P$

$P \succeq 0$ $P \ne 0$ $tr(P^TA) = tr(P^TUSU^T) = tr(U^TP^TUS) = tr(Q^TS)$

$Q = UPU^T$

hold x fixed $\min_Q tr(Q^TS)$ st $tr(Q) \le 1 - x^Tx$ $Q \succeq 0$

$Q = \begin{pmatrix} 0 & & 0 \\ & k & \\ 0 & & 0 \end{pmatrix} \Rightarrow P = k u_i u_i^T$ assume wlog $u_i^Tb \le 0$

$x \to x + \sqrt{k}\, u_i$ better $\longrightarrow$ contradiction

$P \to 0$