Separating hyperplanes

- \( S \) a closed, convex set
- Point \( x \) not in \( S \)
- \( \implies \) strict separating hyperplane

- Suppose \( S, T \) two closed convex sets
- Can they be strictly separated?
Example
Intersection and union

• \((K_1 \cup K_2)^* = \)

• \((K_3 \cap K_4)^* = \)
Flat, pointed, solid, proper

• K is **flat** if:
  • E.g., K =

• K is **pointed** if:
  • E.g., K =

• K is **proper** if:
  • E.g., K =
Generalized inequalities

• Given proper cone $K$
• $x \geq_K y$ iff $x - y \geq_K 0$ iff

• $x >_K y$ iff $x \geq_K y$ and $x \neq y$
• $x \leq_K y$ and $x <_K y$: as expected
• Transitive:
• Examples:
Dual sets

• Any convex set \( C \)
  – e.g.,
• can be represented as intersection of
  – a convex cone:
  – and the hyperplane:
• Dual set: \( C^* = \)
For example

- Dual of unit sphere
Equivalent definition

$$C^* = \{ y \mid$$
More examples

- \{ x \mid x^T A x \leq 1 \} \quad A \text{ invertible}

- Unit square \{ (x, y) \mid -1 \leq x, y \leq 1 \}
Cuboctahedron
Voronoi diagram

• Given points $x_i \in \mathbb{R}^n$
• Voronoi region for $x_i$: 
Properties of dual sets

- Face of set $\iff$ corner of dual
- Corner of set $\iff$ face of dual
- $A \quad B \quad A^* \quad B^*$
- $A^*$ is closed and convex
- $A^{**} = A$ if
- $(A \cap B)^* =$
Duality of norms

• Dual norm definition
  \[ \|y\|_* = \max \]

• Motivation: Holder’s inequality
  \[ x^T y \leq \|x\| \|y\|_* \]
Dual norm examples
Dual norm examples
Dual norm examples
\[ \|y\|_\ast \text{ is a norm} \]

- \[ \|y\|_\ast \geq 0: \]

- \[ \|ky\|_\ast = |k| \|y\|_\ast: \]

- \[ \|y\|_\ast = 0 \text{ iff } y = 0: \]

- \[ \|y_1 + y_2\|_\ast \leq \|y_1\|_\ast + \|y_2\|_\ast \]
Dual-norm balls

- \{ y \mid \|y\|_* \leq 1 \} =

- Duality of norms:
Dual functions

- Arbitrary function $F(x)$
- Dual is $F^*(y) =$

- For example: $F(x) = x^T x / 2$

- $F^*(y) =$
Examples

• $\frac{1}{2} - \ln(-x)$

• $e^x$

• $x \ln(x) - x$
Examples

• $ax + b$:

• $l_K(x)$, cone $K$:

• $l_C(x)$, set $C$:
Examples

• $F(x) = x^TQx, \ Q \ psd$: 

• $F(X) = -\ln |X|, \ X \ psd$: 

• $F(x) = \|x\|^2/2$