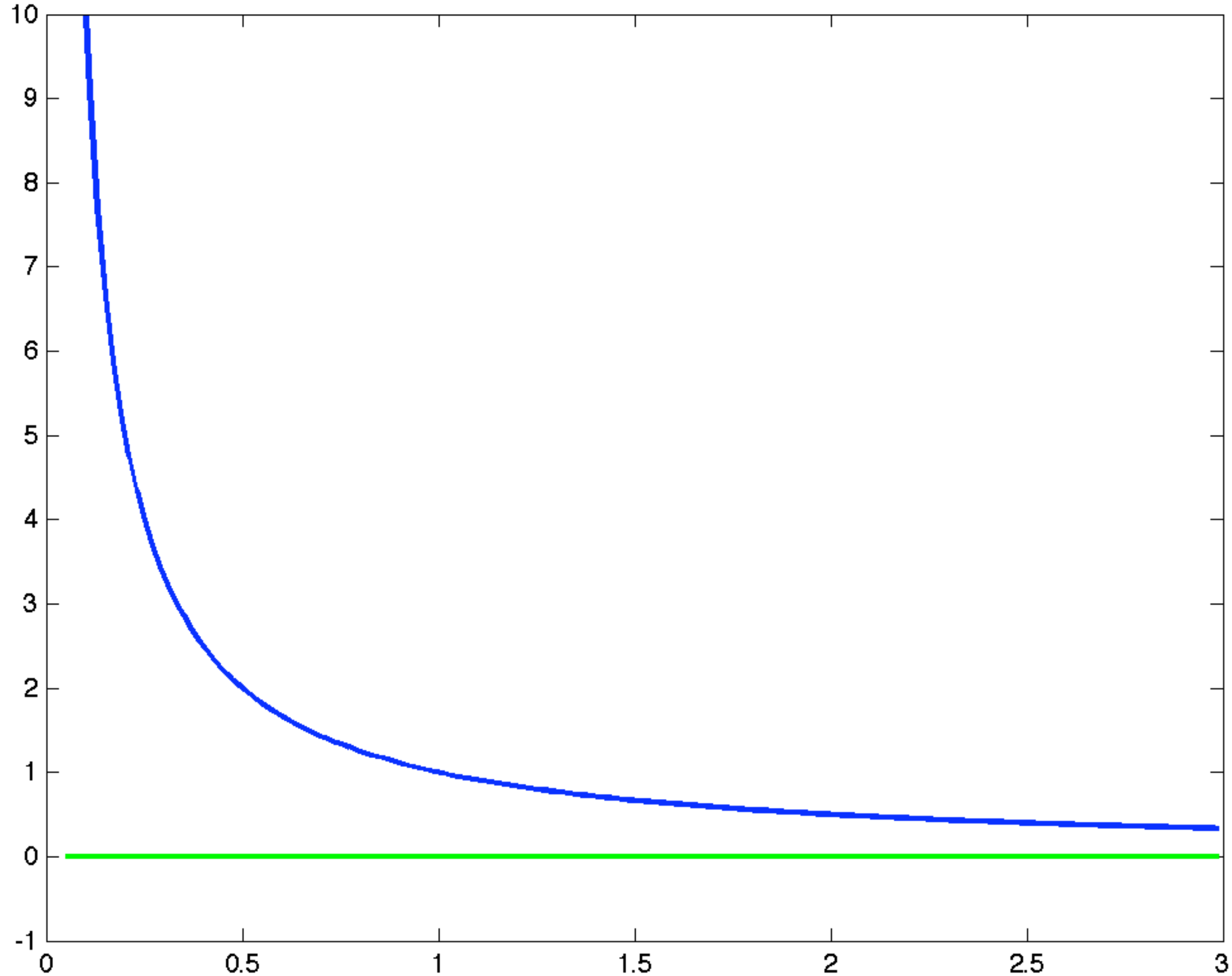
Separating hyperplanes

- S a closed, convex set
- Point x not in S
- ==> strict separating hyperplane

- Suppose S, T two closed convex sets
- Can they be strictly separated?

Example



Intersection and union

- $(K_1 \cup K_2)^* =$

- $(K_3 \cap K_4)^* =$

Flat, pointed, solid, proper

- K is **flat** if:
- E.g., K =
- K is **pointed** if:
- E.g., K =
- K is **proper** if:
- E.g., K =

Generalized inequalities

- Given proper cone K
- $x \geq_K y$ iff $x - y \geq_K 0$ iff

- $x >_K y$ iff $x \geq_K y$ and x != y
- $x \leq_K y$ and $x <_K y$: as expected
- Transitive:
- Examples:

Dual sets

- Any convex set C
  - e.g.,
- can be represented as intersection of
  - a convex cone:
  - and the hyperplane:
- Dual set: C* =

For example

- Dual of unit sphere

Equivalent definition

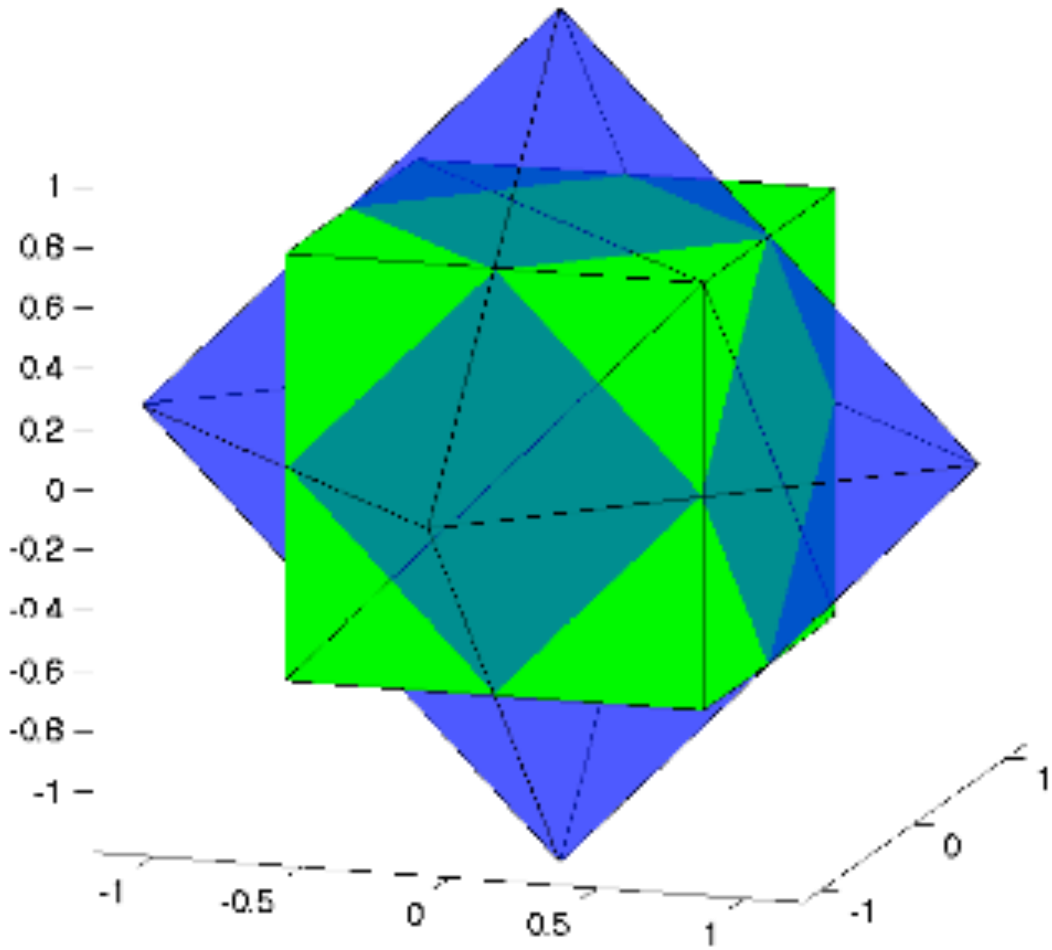
$C^* = \{ y \mid$

More examples

- $\{ x \mid x^T A x \leq 1 \}$ A invertible

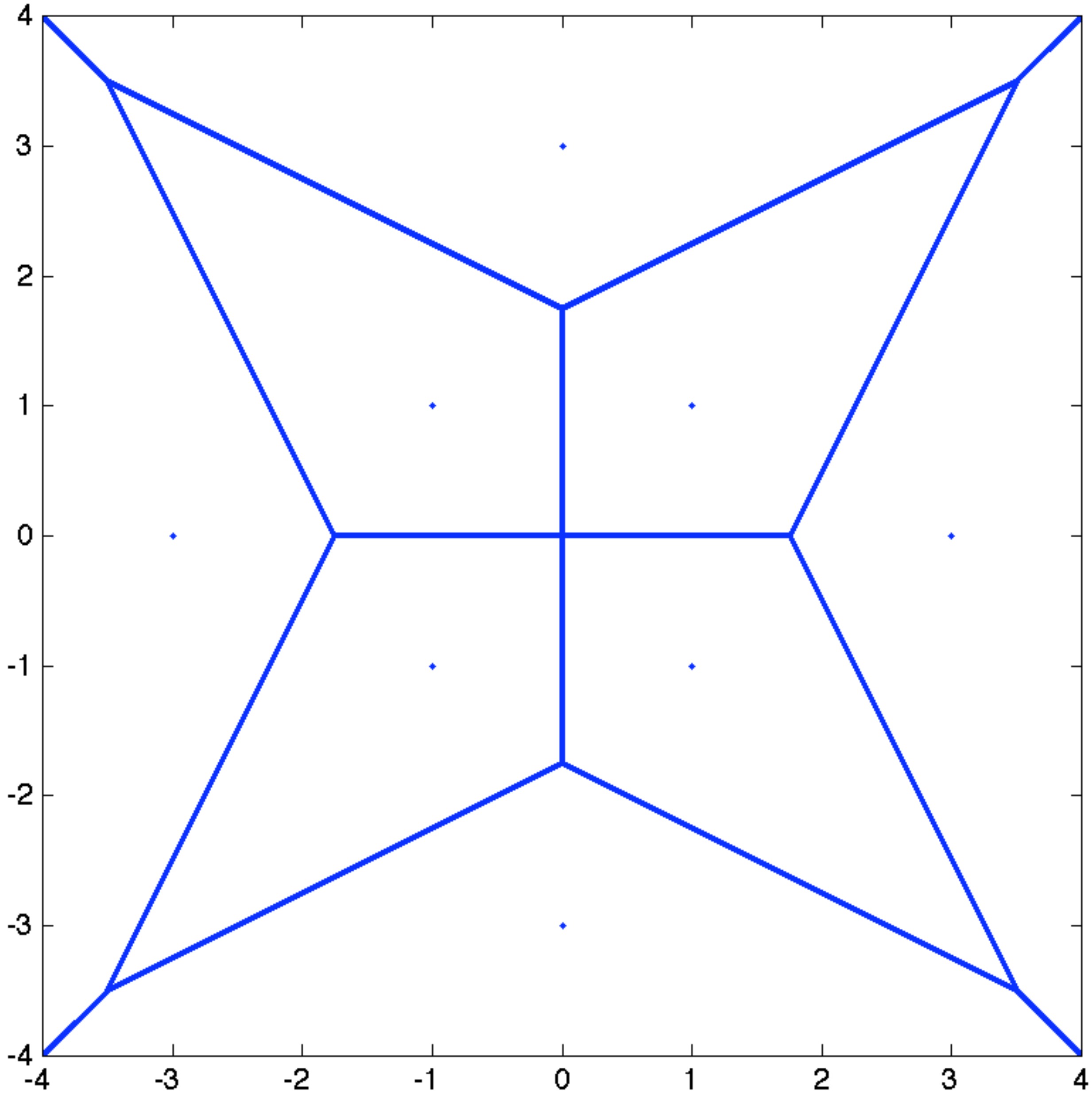
- Unit square $\{ (x, y) \mid -1 \leq x,y \leq 1 \}$

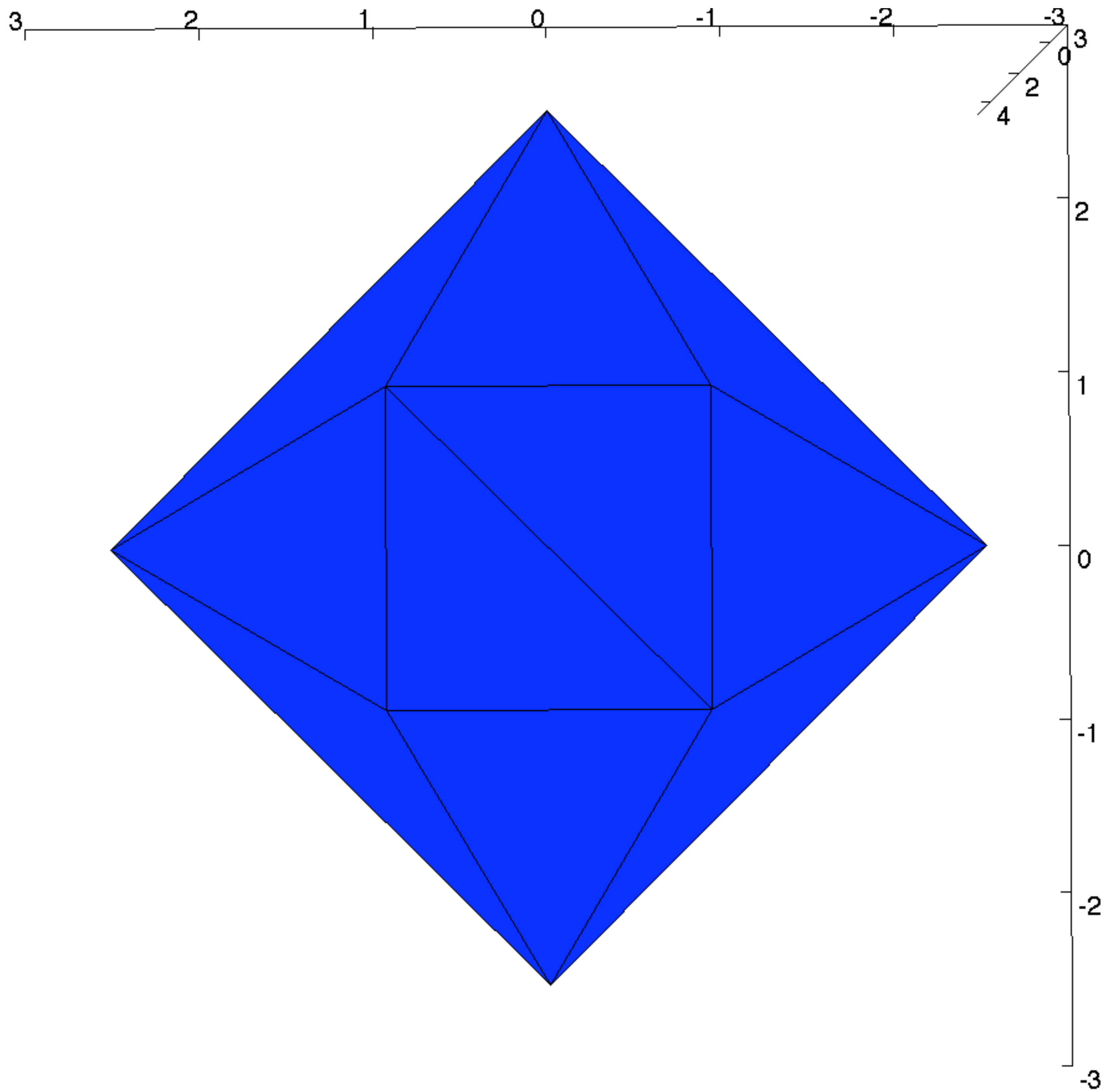
Cuboctahedron



Voronoi diagram

- Given points $x_i \in R^n$
- Voronoi region for x_i:


4
3
2
1
0
-1
-2
-3
-4
-4
-3
-2
-1
0
1
2
3
4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2
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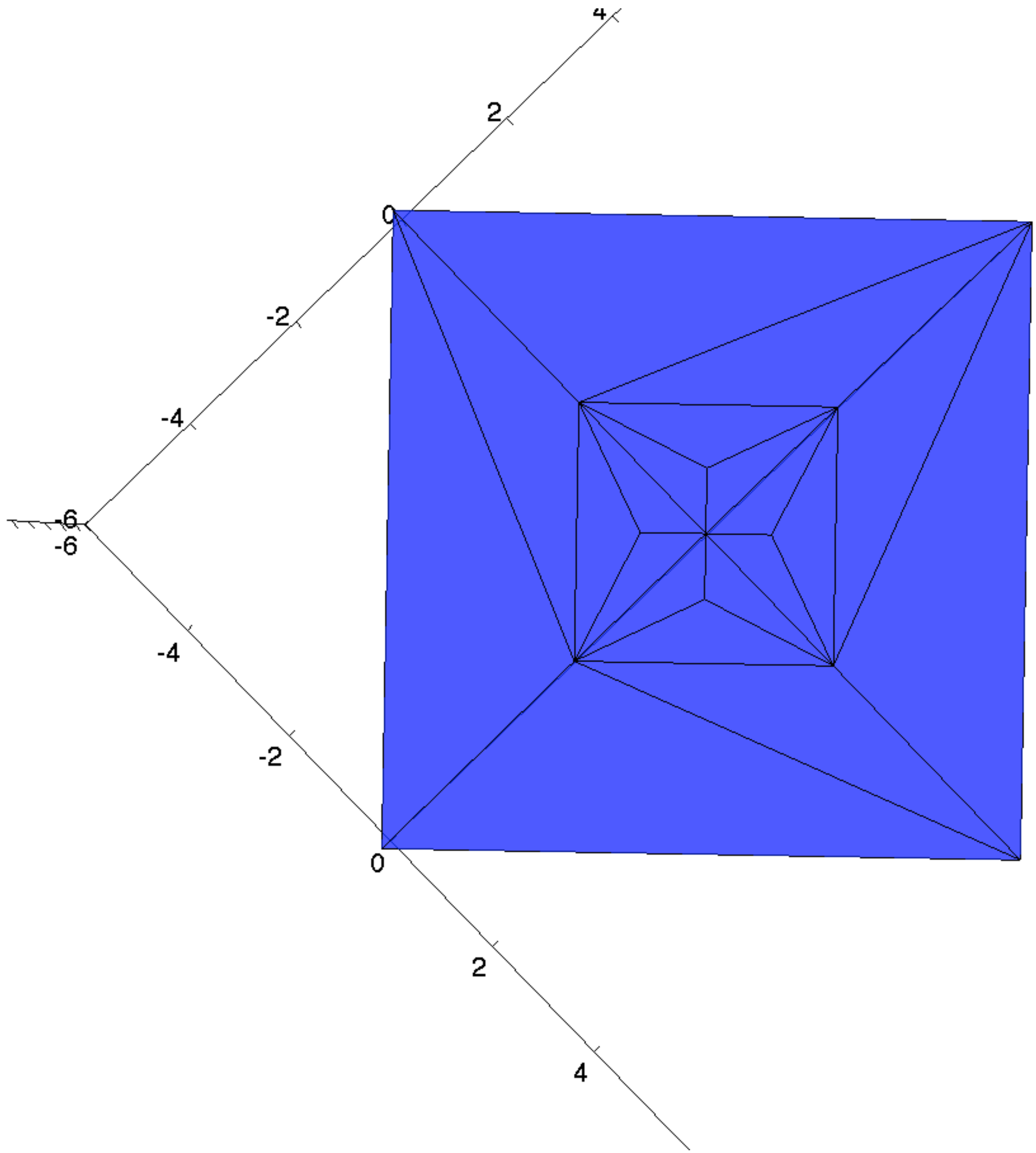
-1
-2
-3



4
2
0
-2
-4
-6
-6
-4
-2
0
2
4

Properties of dual sets

- Face of set <==> corner of dual
- Corner of set <==> face of dual
- A $\quad$ B $\qquad$ A^* $\quad$ B^*
- A^* is closed and convex
- $A^{**} = A$ if
- $(A \cap B)^* =$

Duality of norms

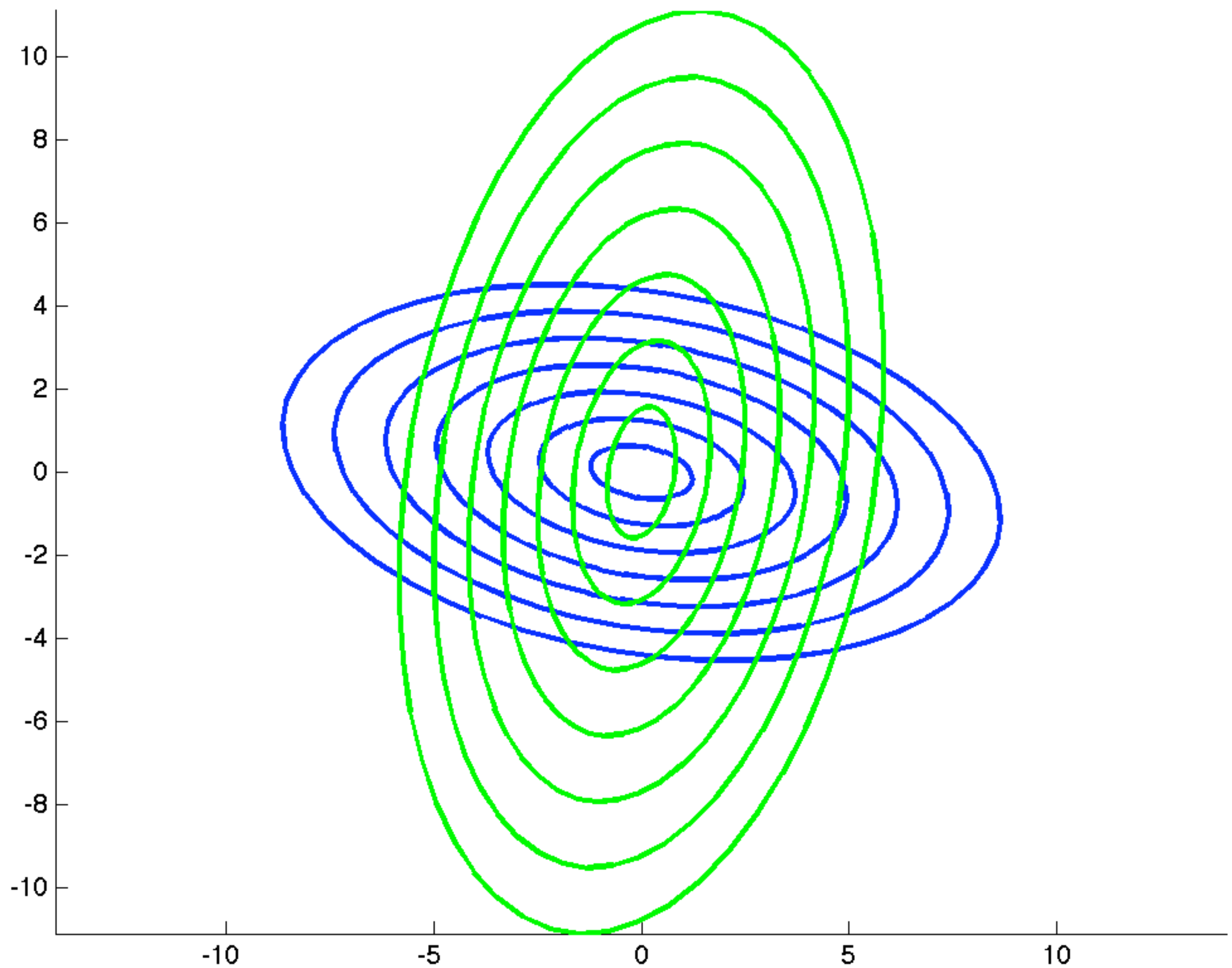
- Dual norm definition

  $||y||_* = \max$

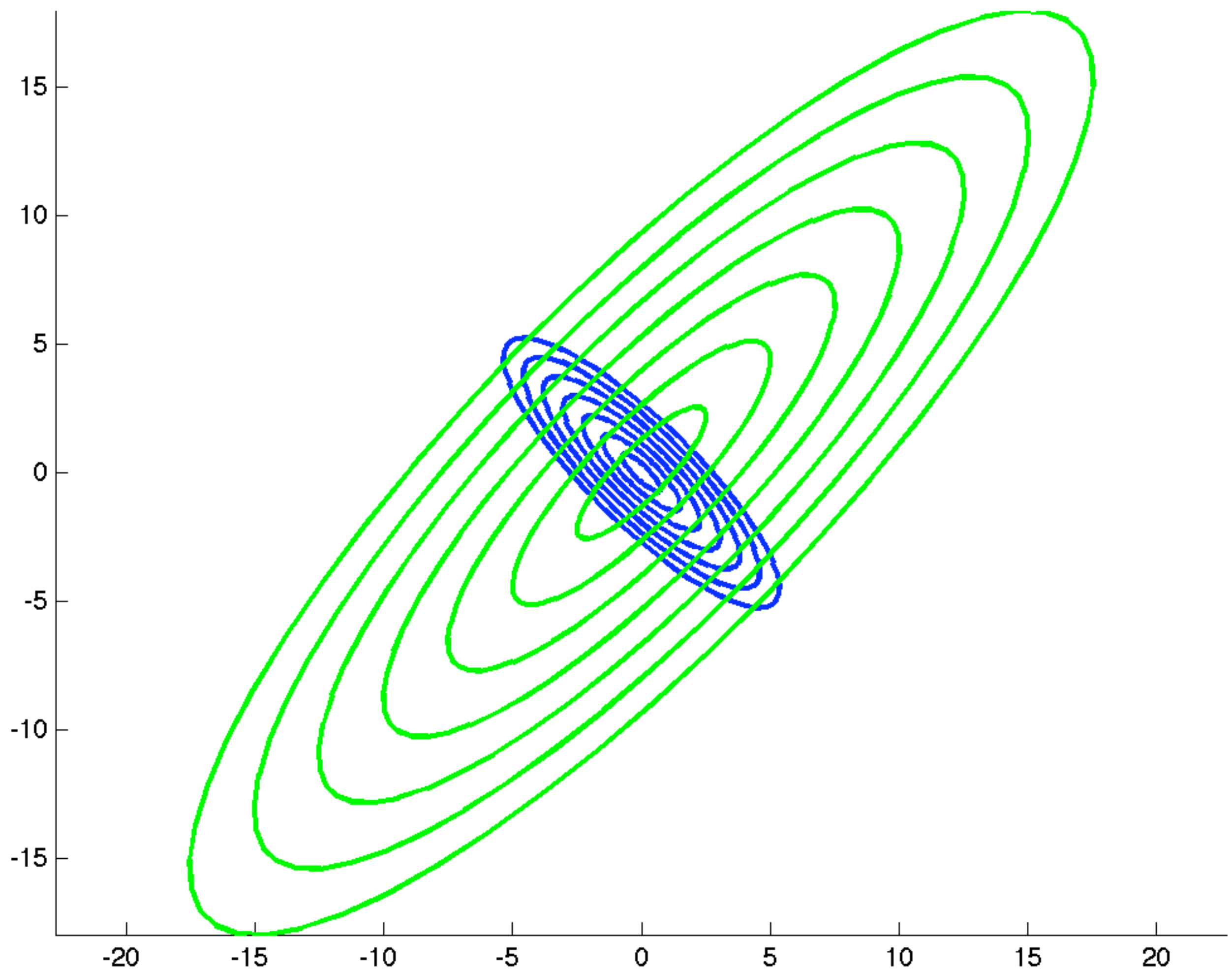
- Motivation: Holder's inequality

  $x^T y \leq ||x|| \, ||y||_*$

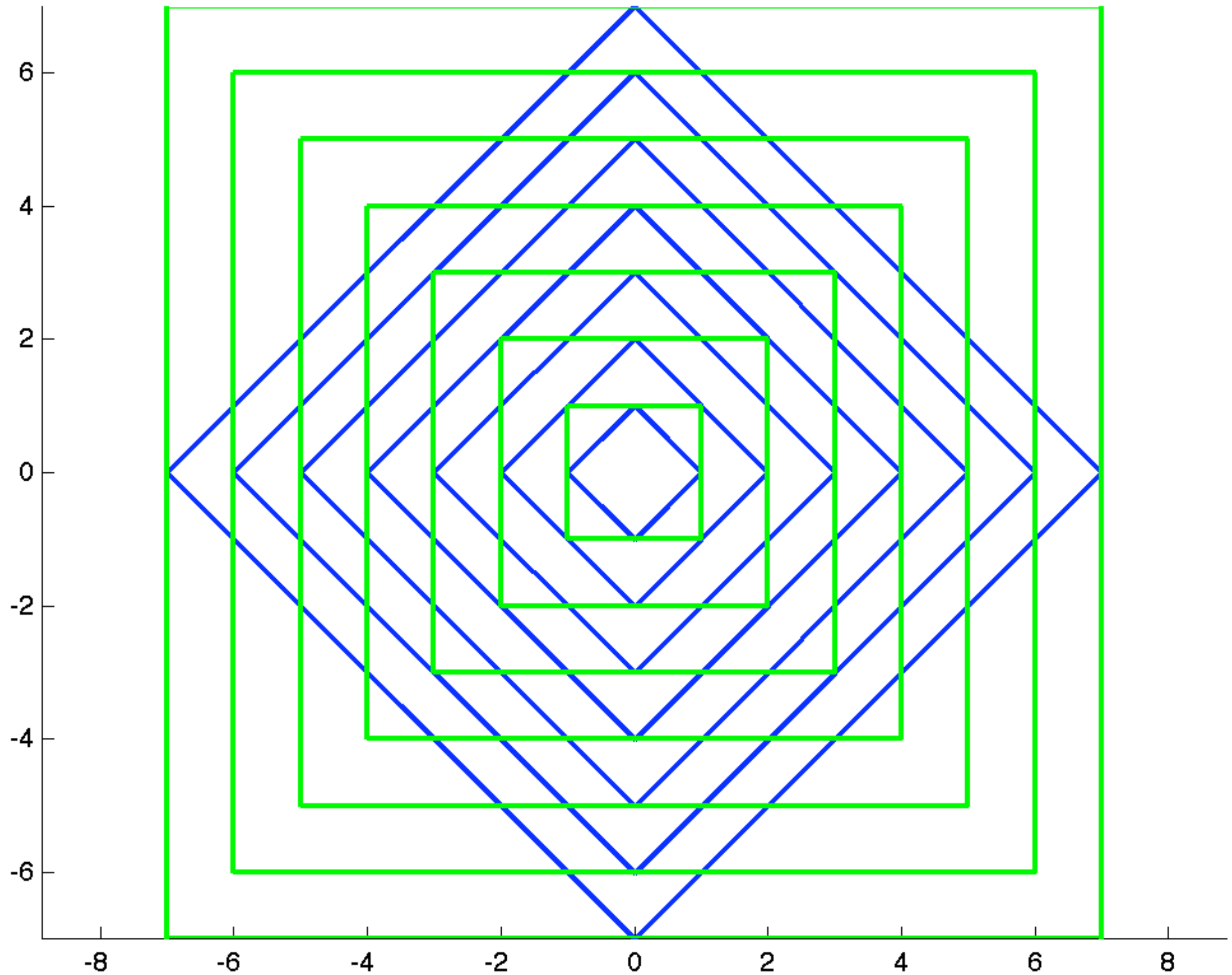
Dual norm examples



Dual norm examples



Dual norm examples



$||y||_*$ is a norm

- $||y||_* \geq 0$:

- $||ky||_* = |k|\ ||y||_*$:

- $||y||_* = 0$ iff $y = 0$:

- $||y_1+y_2||_* \leq ||y_1||_* + ||y_2||_*$

Dual-norm balls

- $\{ y \mid \|y\|_* \leq 1 \} =$

- Duality of norms:

Dual functions

- Arbitrary function $F(x)$
- Dual is $F^*(y) =$

- For example: $F(x) = x^T x/2$

- $F^*(y) =$

Examples

- $1/2 - \ln(-x)$
- e^x
- $x \ln(x) - x$

Examples

- $ax + b$:

- $I_K(x)$, cone K:

- $I_C(x)$, set C:

Examples

- $F(x) = x^T Q x$, Q psd:

- $F(X) = -\ln |X|$, X psd:

- $F(x) = ||x||^2/2$