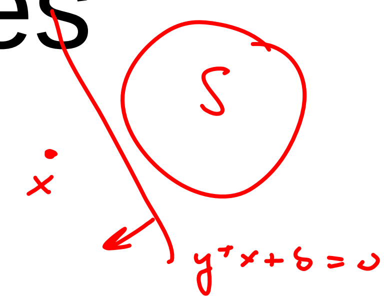
Separating hyperplanes



- S a closed, convex set
- Point x not in S
- ==> strict separating hyperplane

$\exists y, b.\ y^T x + b < 0 \ \wedge$

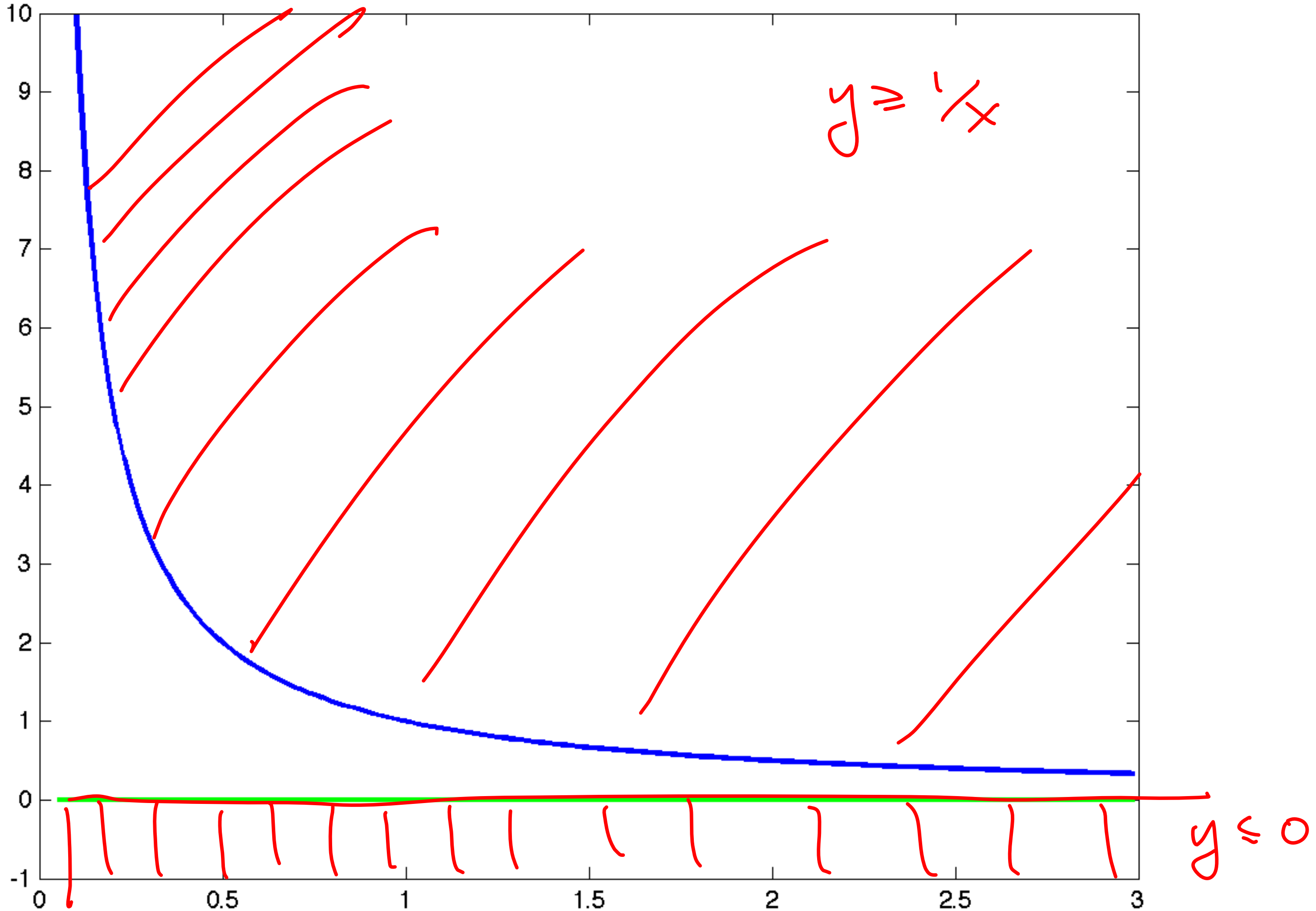
$y^T s + b > 0 \quad \forall s \in S$

- Suppose S, T two closed convex sets
- Can they be strictly separated?

not intersecting

$S \cap T = \emptyset$

Example



$K^* = \{ y \mid y^T x \geq 0 \ \forall x \in K \}$

Intersection and union

- $(K_1 \cup K_2)^* = K_1^* \cap K_2^*$

$y \in K_1^* \cap K_2^* \iff (y^T x \geq 0 \ \forall x \in K_1) \wedge (y^T x \geq 0 \ \forall x \in K_2)$

$y^T x \geq 0 \ \forall x \in K_1 \cup K_2 \iff y \in (K_1 \cup K_2)^*$

$(K_1 \cup K_2)^{**} = (K_1^* \cap K_2^*)^*$ Define $K_3 = K_1^*$ $K_4 = K_2^*$

$\text{cl conv}(K_1 \cup K_2) = (K_3 \cap K_4)^*$ K_3 and K_4 closed & convex

- $(K_3 \cap K_4)^* =$ $\text{cl conv}(\text{cl conv } K_1 \cup \text{cl conv } K_2)$

$(K_1^*)^* = K_3^*$ $(K_2^*)^* = K_4^*$

$= \text{cl conv}(K_3^* \cup K_4^*)$

Flat, pointed, solid, proper

K is a convex cone

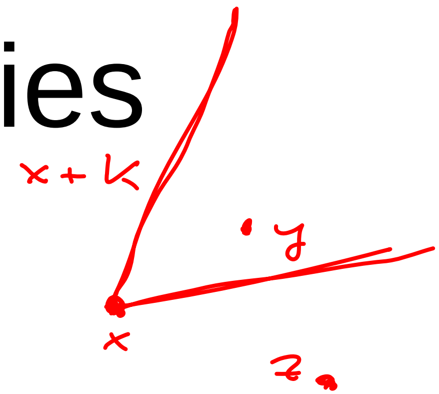
- K is **flat** if: $\exists x \neq 0 . \forall \lambda \in \mathbb{R} . \lambda x \in K$
- E.g., K = $\{(x,y,z) \mid x \geq 0, y \geq 0\}$ $(1,1,z) \in K$ $\forall z$
- K is **pointed** if: not flat
- E.g., K = $\{(x,y,z) \mid x = 0, y, z \geq 0\}$ ← improper
- K is **proper** if: K, K^* both pointed
- E.g., K = $\{(x,y,z) \mid x \geq 0, y \geq 0, z \geq 0\}$

$\Rightarrow \text{int } K \neq \emptyset \quad \text{int } K^* \neq \emptyset$

if K improper, could have $x <_K y \quad y <_K x$

Generalized inequalities

$y \in x+K \iff y - x \in K$



- Given proper cone K
- $x \geq_K y$ iff $x - y \geq_K 0$ iff $x - y \in K$

$y \geq_K x$

$z \not\geq_K x$

$\Rightarrow \quad x \geq_K y \ \wedge \ y \geq_K z \qquad x - y \in K,\ y - z \in K \qquad x - y + y - z \in K,\ x - z \in K$

- $x >_K y$ iff $x \geq_K y$ and $x \mathrel{!=} y$
- $x \leq_K y$ and $x <_K y$: as expected
- Transitive:
- Examples:

$x \geq_{\mathbb{R}^n_+} y \iff x_i \geq y_i \quad \forall i$

$x \geq_{S_+} y \iff x - y$ positive semidefinite

$\iff x \succeq y$

Dual sets

Arbitrary C:

$K = \{(x, s) \mid x \in sC,\ s \geq 0\}$

- Any convex set C
  - e.g., $\{x \mid Ax + b \geq 0\}$ $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$
- can be represented as intersection of
  - a convex cone: $\{(x, s) \mid Ax + bs \geq 0\}$ $s \in \mathbb{R}$, $s \geq 0$

    $K =$

    $A\lambda x + b\lambda s = \lambda(Ax + bs) \geq 0$
  - and the hyperplane: $\{(x, s) \mid s = 1\}$ $\{(x, 1) \mid Ax + b \geq 0\}$
- Dual set: C* = $-K^* \cap \{(y, t) \mid t = -1\}$

For example

- Dual of unit sphere

$B = \{x \mid \|x\| \le 1\}$

$s=1$

$SOC = \{(x,s) \mid \|x\| \le s\}$

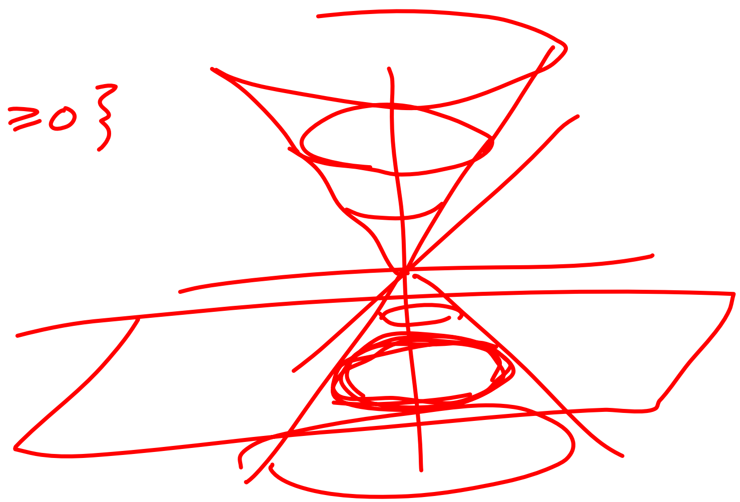
$SOC \cap \{(x,s) \mid s=1\} = \{(x,s) \mid \|x\| \le 1\} = B$

$B^* = \{-\underbrace{SOC^*}_{SOC} \cap \{(y,t) \mid t=-1\} = \{\|y\| \le 1\} = B$

$-SOC = \{(-y,-t') \mid \|y\| \le t', t' \ge 0\}$

$-t' = -1$

$\{(-y,-1) \mid \|y\| \le 1\}$



Equivalent definition

$C^* = \{\, y \mid y^T x \le 1 \quad \forall x \in C \,\}$

$K = \{ (x, s) \mid x \in sC, \; s \ge 0 \}$

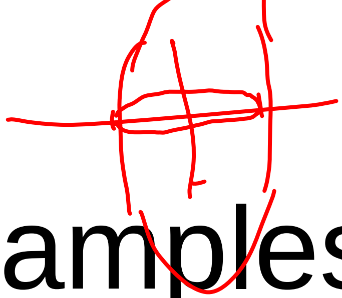
$K^* = \{ (y, t) \mid x^T y + st \ge 0 \quad \forall (x, s) \in K \}$

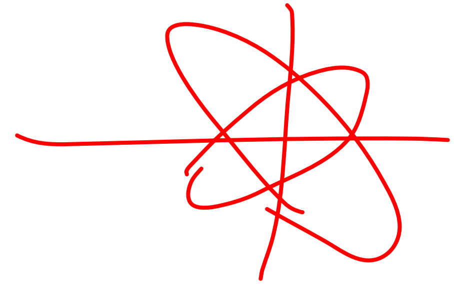
$-K^* = \{ (-y', -t') \mid x^T y' + st' \ge 0 \quad \forall (x, s) \in K \}$

$-t' = -1$

$x^T y' + s$

More examples





- $\{ x \mid x^T A x \le 1 \}$ A invertible, $A \succeq 0$

$\rightarrow \{ y \mid y^T A^{-1} y \le 1 \}$

$A = U^T U$ U invertible

$z = Ux$

$x^T A x \le 1 \iff x^T U^T U x \le 1$

$\iff z^T z \le 1$

$w = U^{-1} y$

$y^T A^{-1} y \le 1$

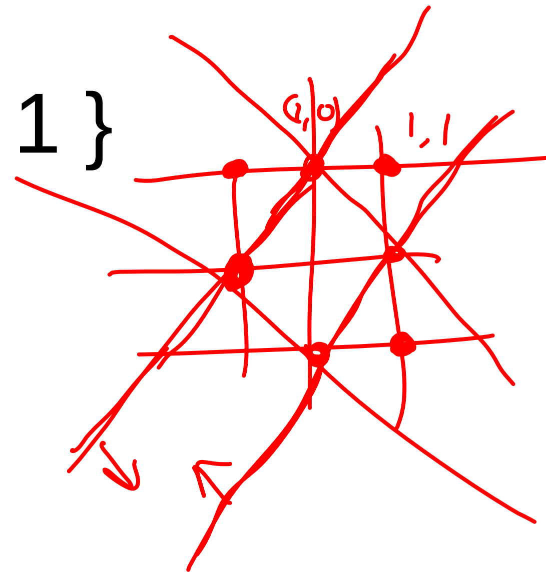
$\iff w^T w \le 1$

$x^T y = (U^{-1} z)^T (U w)$

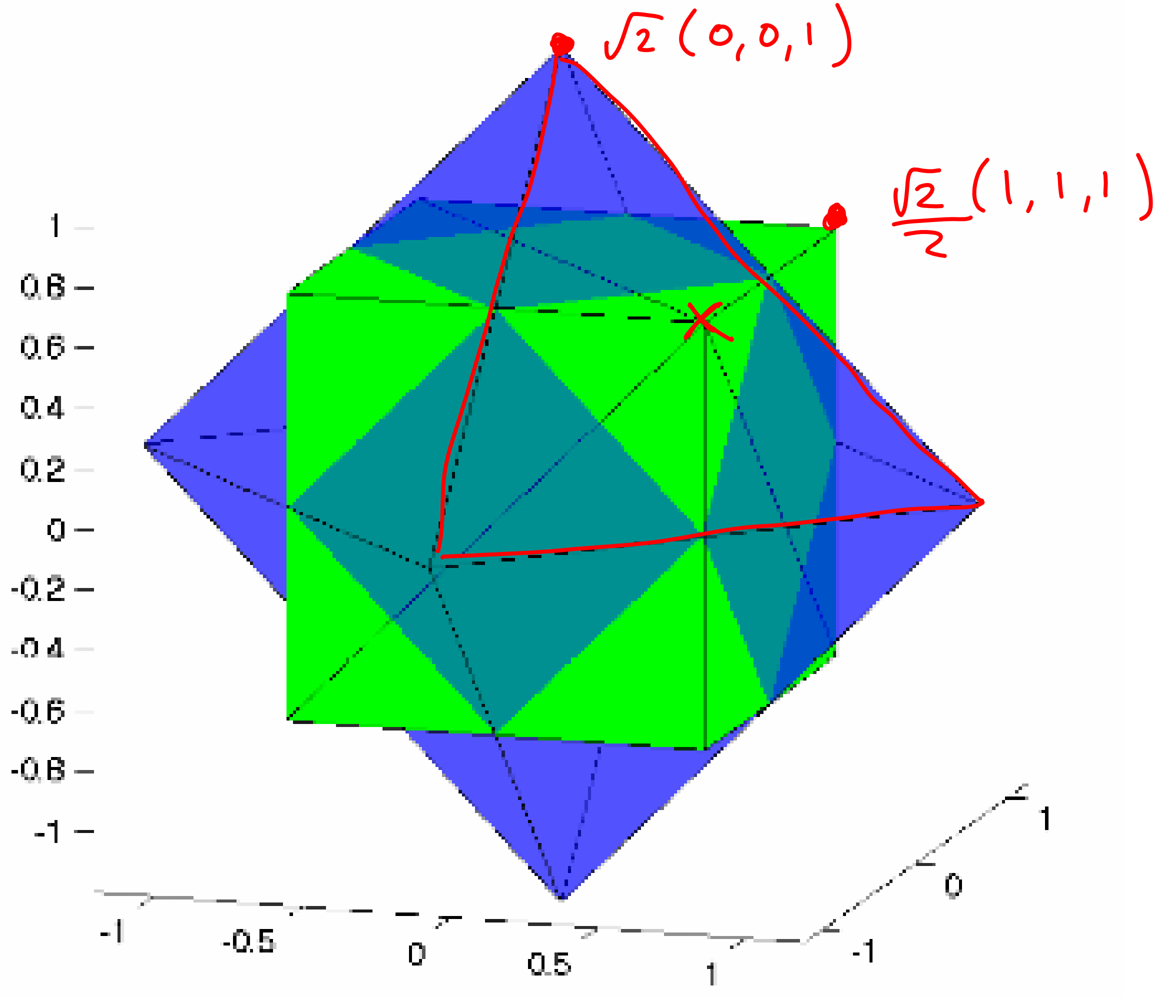
$= z^T w$

$\le \|z\| \, \|w\| \le 1$

- Unit square $\{ (x, y) \mid -1 \le x,y \le 1 \}$



Cuboctahedron



Voronoi diagram

- Given points $x_i \in R^n$ $x_1 \ldots x_k$
- Voronoi region for x_i: $\{x \mid \|x - x_i\| \leq \|x - x_j\| \ \forall j \neq i\}$

$\Rightarrow \quad y_i = \begin{pmatrix} x_i \\ \|x_i\|^2/2 \end{pmatrix} \in \mathbb{R}^{n+1} \quad S = \{y_i\}$ has k elements

$\Rightarrow \quad S^* = \{(z,t) \mid z^T x_i + t\|x_i\|^2/2 \leq 1 \ \forall i\}$

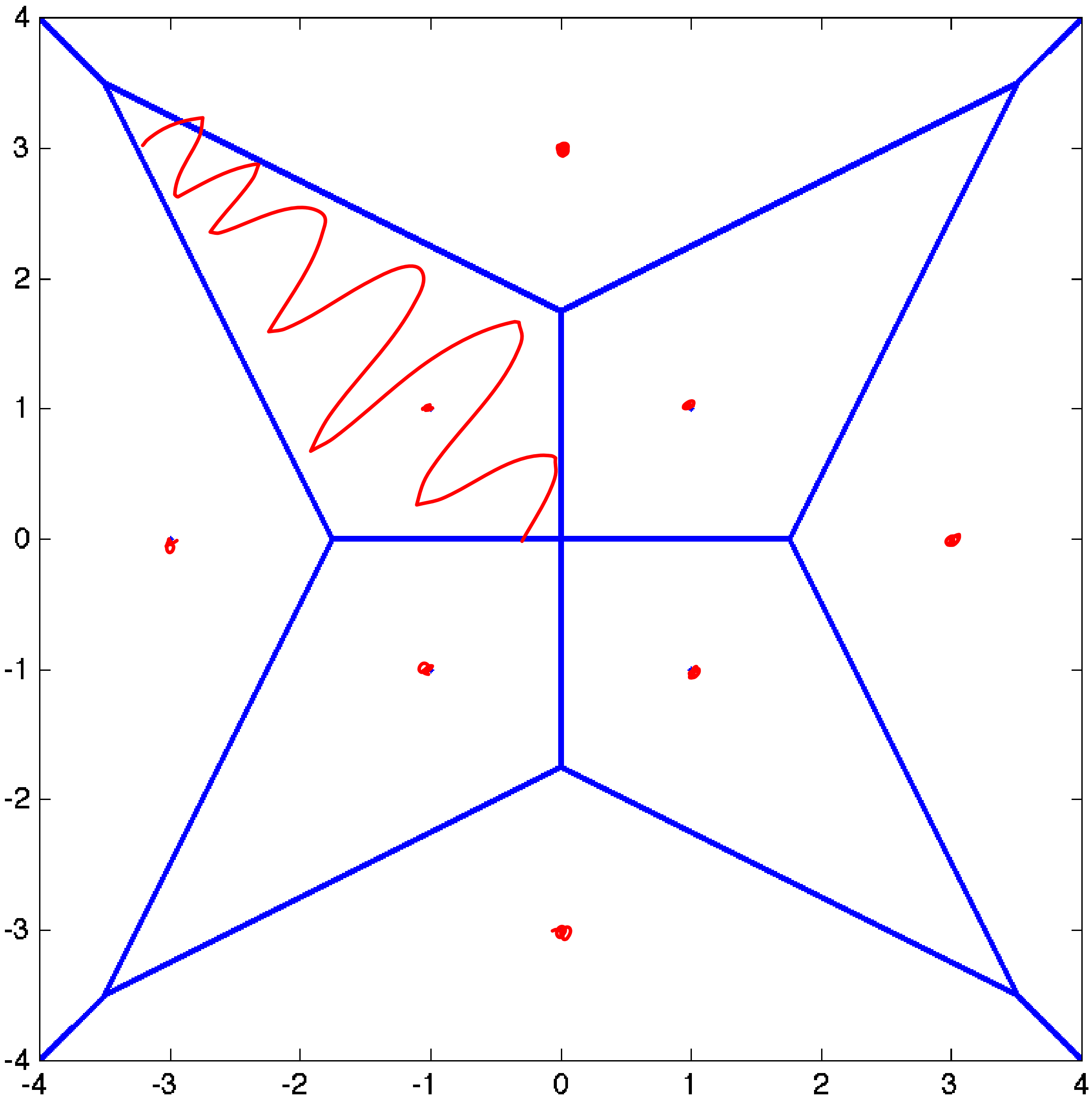
face i active $\Leftrightarrow z^T x_i + t\|x_i\|^2/2 \leq z^T x_j + t\|x_j\|^2/2 \quad \forall j \neq i$

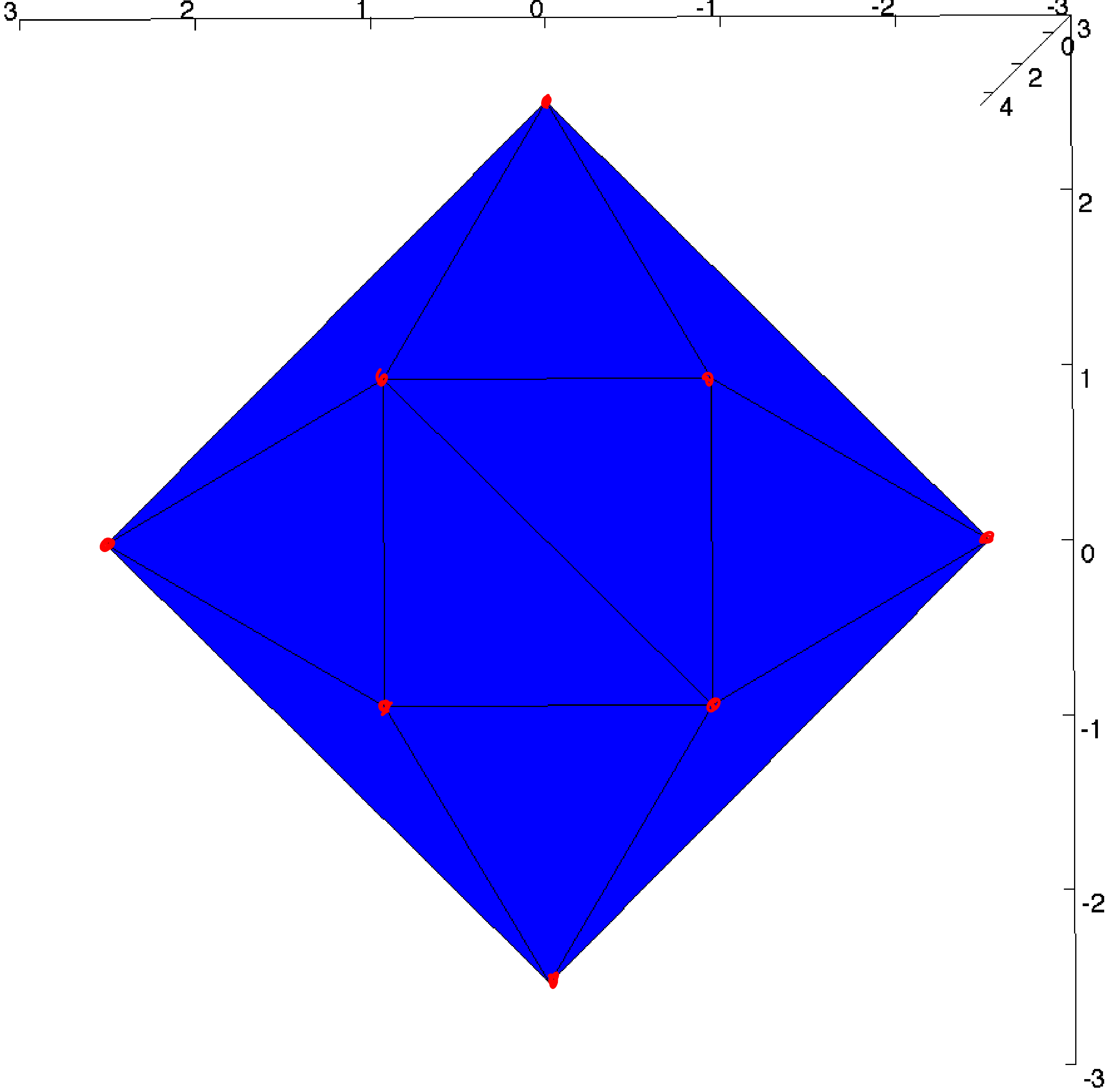
$\Leftrightarrow \quad \frac{z}{t}^T x_i + \|x_i\|^2/2 + \|\tfrac{z}{t}\|^2/2 \leq \frac{z}{t}^T x_j + \|x_j\|^2/2 + \|z/t\|^2/2$

$y = -z/t$

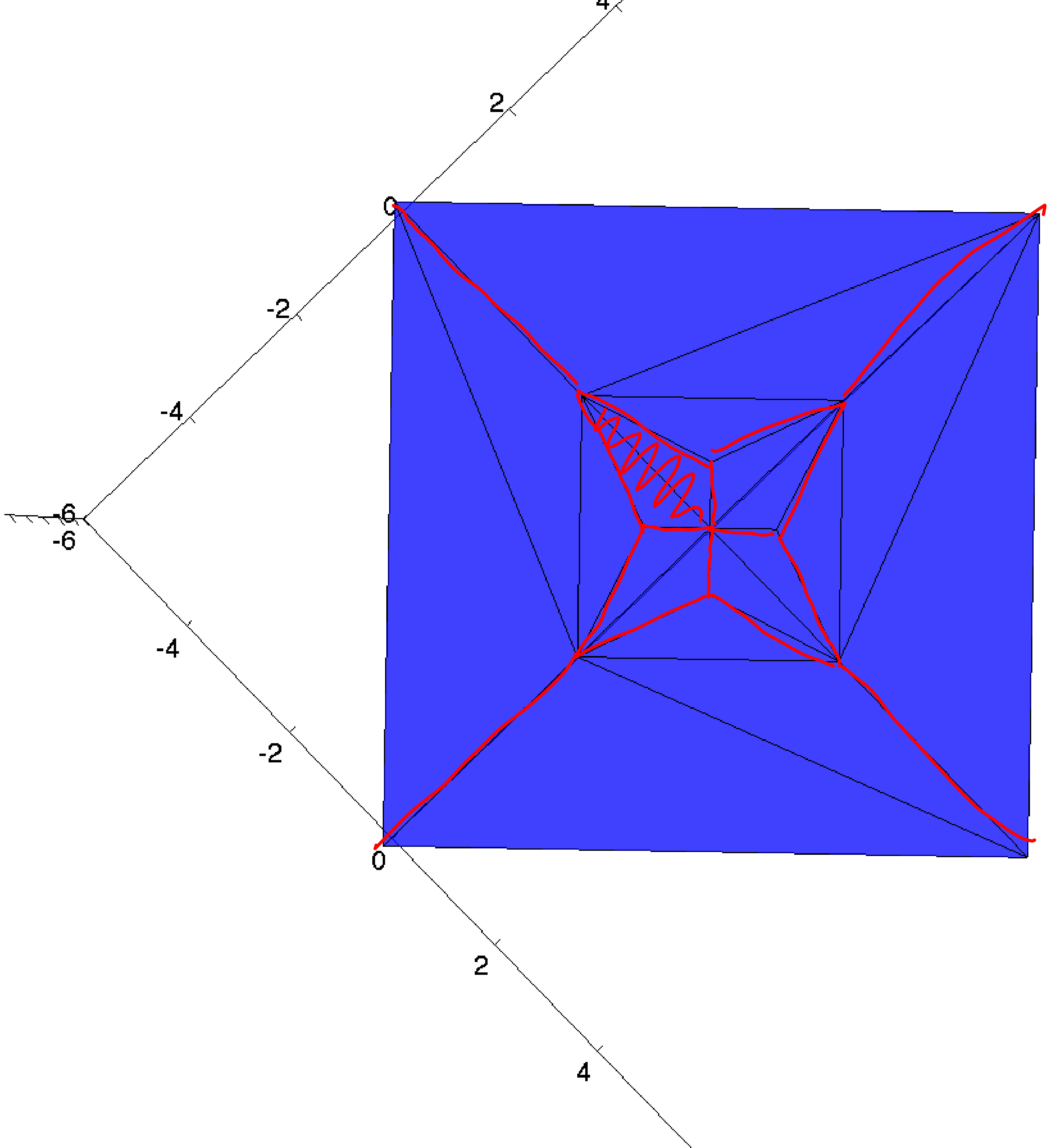
$\Leftrightarrow \quad -y^T x_i + \|x_i\|^2/2 + \|y\|^2/2 \leq -y^T x_j + \|x_j\|^2/2 + \|y\|^2/2$

$\|y - x_i\|^2/2 \leq \|y - x_j\|^2/2$


4
3
2
1
0
-1
-2
-3
-4
-4
-3
-2
-1
0
1
2
3
4



3
2
1
0
-1
-2
-3
3
0
2
4
2
1
0
-1
-2
-3



4
2
0
-2
-4
-6
-6
-4
-2
0
2
4

Properties of dual sets

- Face of set <==> corner of dual
- Corner of set <==> face of dual
- A $\subsetneq$ B $\Leftrightarrow$ A* $\supsetneq$ B*
- A* is closed and convex
- A** = A if A closed, convex
- (A ∩ B)* = conv (A* ∪ B*) if A, B closed, convex