Convex Functions (cont. 2)

Optimization - 10725
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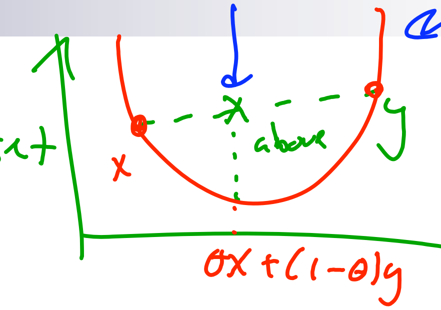
Convex Functions

- Function $f: R^n \rightarrow R$ is convex if
  - Domain is convex — dom f convex set
  - $\forall x, y \in \text{dom} f, \ \theta \in [0,1]$



$$f(\theta x + (1-\theta) y) \leq \theta f(x) + (1-\theta) f(y)$$

- Generalization: Jensen's inequality:

prob dist $x \in$ convex domain of f

$$f[E[x]] \leq E_x[f(x)]$$

useful in ML
e.g., EM

- Strictly convex function:

e.g., ∪ , convex non-strict ／

$\forall x, y \in \text{dom} f, \ \theta \in (0,1)$
$x \neq y$

$$f(\theta x + (1-\theta) y) < \theta f(x) + (1-\theta) f(y)$$

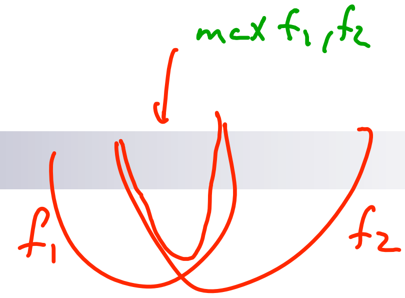
Operations that preserve convexity

- Many operations preserve convexity
  - Knowing them will make your life much easier when you want to show that something is convex
  - Examples in next few slides
- Simplest: Non-negative weighted sum:
  - $f = \sum_i w_i f_i \quad w_i \geq 0$
  - If all f_i's are convex, then f is convex
  - If all f_i's are concave, then f is concave
  - Example: integral of f(x,y) convex x,y , $g(x) = \int_{y \in C} f(x,y)\, dy$ → convex
- Affine mapping: $f:\mathbb{R}^n \rightarrow \mathbb{R}$, $A \in \mathbb{R}^{n \times m}$, $b \in \mathbb{R}^m$ e.g., A, b come from PCA
  - $g(x) = f(Ax+b)$
  - **dom** g = $\{x \mid Ax+b \in \text{dom } f\}$ ← always convex if dom f convex
  - If f is convex, then g is convex
  - If f is concave, then g is concave

Pointwise maximum and supremum

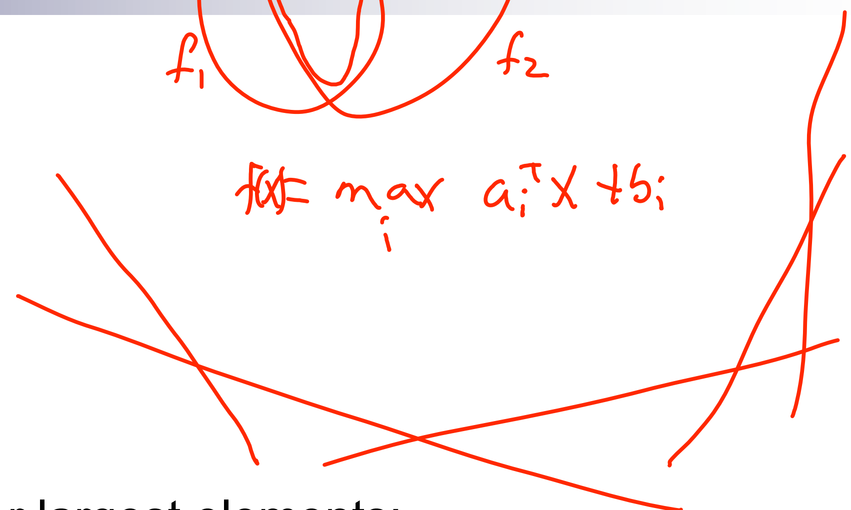
- If f_i's are convex, then

  $f(x) = \max_i f_i(x)$



- Piecewise linear convex functions:
  - Fundamental for POMDPs

  $f(x) = \max_i a_i^T x + b_i$



- For x in a convex set C, sum of the r largest elements:
  - Sort x, pick r largest components, sum them:

  $f(x) = \{ \sum_{i=1}^{r} x_{l_i} \mid x_{l_1} \geq x_{l_2} \geq \dots \geq x_{l_n} \}$

  max of $\binom{n}{r}$ different functions

  $f(x) = \max_{A \subseteq \{1,\dots,n\},\ |A|=r} x_{A_1} + x_{A_2} + \dots + x_{A_r}$

- Maximum eigenvalue of symmetric matrix $X \in \mathbb{R}^{n \times n}$, $f: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$
  - f(X) = $\max_{v,\ \|v\|_2 = 1} v^T X v$ ← max of linear functions of X

Pointwise maximum of affine functions: general representation

- We saw: convex set can be written as intersection of (infinitely many) hyperplanes:
  - C convex, then
- Convex functions can be written as supremum of (infinitely many) lower bounding hyperplanes:
  - f convex function, then

Discussion on this slide subject to mild conditions on sets and functions, see book

Composition: scalar differentiable, real domain case

- How do I prove convexity of log-sum-exp-positive-weighted-sum-monomials? :)
  -
- If $h:R^k \rightarrow R$ and $g:R^n \rightarrow R^k$, when is $f(x) = h(g(x))$ convex (concave)?
  - **dom** f = $\{x \in$ **dom** g$|\ g(x) \in$ **dom** h$\}$
- Simple case: $h:R \rightarrow R$ and $g:R^n \rightarrow R$, **dom** g = **dom** h = R, g and h differentiable
  - E.g., $g(x)=x^T \Sigma x$, Σ psd, $h(y) = e^y$
- Second derivative:
  - $f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)$
    - When is $f''(x) \geq 0$ (or $f''(x) \leq 0$) for all x?
- Example of sufficient (but not necessary) conditions:
  - f convex if h is convex and nondecreasing and g is convex
  - f convex if h is convex and nonincreasing and g is concave
  - f concave if h is concave and nondecreasing and g is concave
  - f concave if h is concave and nonincreasing and g is convex

Composition: scalar, general case

- If $h: R^k \rightarrow R$ and $g: R^n \rightarrow R^k$, when is $f(x) = h(g(x))$ convex (concave)?
  - **dom** f = {x ∈ **dom** g| g(x) ∈ **dom** h}

- Simple case: $h: R \rightarrow R$ and $g: R^n \rightarrow R$, general domain and non-differentiable
  - Example of sufficient (but not necessary) conditions:
    - f convex if h is convex and $\tilde{h}$ nondecreasing and g is convex
    - f convex if h is convex and $\tilde{h}$ nonincreasing and g is concave
    - f concave if h is concave and $\tilde{h}$ nondecreasing and g is concave
    - f concave if h is concave and $\tilde{h}$ nonincreasing and g is convex

- nondecreasing or nonincreasing condition on extend value extension of h is fundamental
  - counter example in the book if nondecreasing property holds for h but not for $\tilde{h}$, the composition no longer convex
  - If $h(x)=x^{3/2}$ with **dom** h = R_+, convex but extension is not nondecreasing
  - If $h(x)=x^{3/2}$ for $x \geq 0$, and $h(x)=0$ for $x<0$, **dom** h = R, convex and extension is nondecreasing

Vector composition: differentiable

- If $h: R^k \rightarrow R$ and $g: R^n \rightarrow R^k$, when is $f(x) = h(g(x))$ convex (concave)?
  - **dom** f = {x ∈ **dom** g| g(x) ∈ **dom** h}
- Focus on $f(x) = h(g(x)) = h(g_1(x), g_2(x),\ldots, g_k(x))$
- Second derivative:
  - $f''(x) = g'(x)^T \nabla^2 h(g(x)) g'(x) + \nabla h(g(x))\, g''(x)$
    - When is $f''(x) \geq 0$ (or $f''(x) \leq 0$) for all x?
- Example of sufficient (but not necessary) conditions:
  - f convex if h is convex and nondecreasing in each argument, and g_i are convex
  - f convex if h is convex and nonincreasing in each argument, and g_i are concave
  - f concave if h is concave and nondecreasing in each argument, and g_i are concave
  - f concave if h is concave and nonincreasing in each argument, and g_i are convex
- Back to log-sum-exp-positive-weighted-sum-monomials
  -
  - **dom** $f = R^n_{++}$, $c_i>0$, $a_i \geq 1$
  - log sum exp convex

Minimization

- If f(x,y) is convex in (x,y) and C is a convex set, then:

- Norm is convex: ||x-y||
  - minimum distance to a set C is convex:

Perspective function

- If f is convex (concave), then the perspective of f is convex (concave):
  - $t>0$, $g(x,t) = t\,f(x/t)$
- KL divergence:
  - $f(x) = -\log x$ is convex
  - Take the perspective:
  - Sum over many pairs (x_i, t_i)

Quasiconvex functions

- Unimodal functions are not always convex
- But they are (usually) still easy to optimize: Quasiconvex function:
  - All sublevel sets are convex, for all $\alpha \in R$:
- Equivalent definition: max of extremes is higher than function
- Applications include computer vision (geometric reconstruction) [Ke & Kanade '05]

Log-convex functions

- Function f:$R^n \rightarrow R$, with f(x)>0 in all (convex) **dom** f
  - f log-convex if and only if:
- Or equivalently:
- Examples
  - Gaussian $f(x) = \frac{1}{\sqrt{(2\pi)^n det\Sigma}} e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)}$

What should you know: Convex fns

- definition
- showing that a function is convex/concave
  - first principle
  - first and second order condition
  - epigraph
  - operations that preserve convexity
- quasiconvexity
- log-convexity