Readings:
K&F: 4.1, 4.2, 4.3, 4.4, 4.5

Undirected Graphical Models

Graphical Models – 10708
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Normalization for computing probabilities

- To compute actual probabilities, must compute normalization constant (also called partition function)

$P(ABCD) = \frac{1}{Z} \phi_1(AB)\, \phi_2(BC)\, \phi_3(CD)\, \phi_4(DA)$

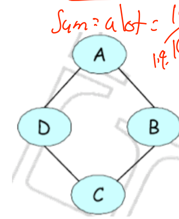
$Z = \sum_a \sum_b \sum_c \sum_d \phi_1(ab)\, \phi_2(bc)\, \phi_3(cd)\, \phi_4(da)$

Assignment				Unnormalized (Potential)	Normalized
a^0	b^0	c^0	d^0	300000	0.04
a^0	b^0	c^0	d^1	300000	0.04
a^0	b^0	c^1	d^0	300000	0.04
a^0	b^0	c^1	d^1	30	$4.1 \cdot 10^{-6}$
a^0	b^1	c^0	d^0	500	$6.9 \cdot 10^{-5}$
a^0	b^1	c^0	d^1	500	$6.9 \cdot 10^{-5}$
a^0	b^1	c^1	d^0	5000000 (a lot)	0.69
a^0	b^1	c^1	d^1	500	$6.9 \cdot 10^{-5}$
a^1	b^0	c^0	d^0	100	$1.4 \cdot 10^{-5}$
a^1	b^0	c^0	d^1	1000000	0.14
a^1	b^0	c^1	d^0	100	$1.4 \cdot 10^{-5}$
a^1	b^0	c^1	d^1	100	$1.4 \cdot 10^{-5}$
a^1	b^1	c^0	d^0	10	$1.4 \cdot 10^{-6}$
a^1	b^1	c^0	d^1	100000	0.014
a^1	b^1	c^1	d^0	100000	0.014
a^1	b^1	c^1	d^1	100000	0.014

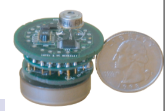
Sum = a lot = $\frac{10}{1.4 \cdot 10^{-6}}$

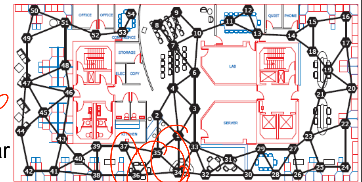
- Computing partition function is hard! → Must sum over all possible assignments

Can use VE to compute Z if Markov Network has low tree width



Factorization in Markov networks





- Given an undirected graph *H* over variables $\mathbf{X}=\{X_1,\ldots,X_n\}$
- A distribution *P* **factorizes** over *H* if ∃
  - subsets of variables $\mathbf{D}_1 \subseteq \mathbf{X},\ldots,\mathbf{D}_m \subseteq \mathbf{X}$, such that the $\mathbf{D}_i$ are *fully connected* in *H*
  - *non-negative potentials* (or factors) $\phi_1(\mathbf{D}_1),\ldots,\phi_m(\mathbf{D}_m)$
    - also known as clique potentials
  - such that

$$P(X) = \frac{1}{Z}\prod_{i=1}^{m}\phi_i(D_i)$$

D_i, D_j may overlap

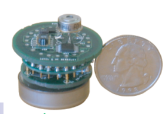
$D_1 = \{1, 34, 35\}$

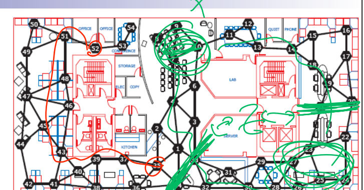
$D_2 = \{36, 37\}$

$D_3 = \{34, 35, 36\}$

- Also called Markov random field *H*, or Gibbs distribution over *H*

Global Markov assumption in Markov networks





- A path $X_1 - \ldots - X_k$ is **active** when set of variables **Z** are observed if none of $X_i \in \{X_1,\ldots,X_k\}$ are observed (are part of **Z**)
- Variables **X** are **separated** from **Y** given **Z** in graph *H*, $sep_H(\mathbf{X};\mathbf{Y}|\mathbf{Z})$, if there is no active path between any $X \in \mathbf{X}$ and any $Y \in \mathbf{Y}$ given **Z**

$sep_H(X;Y|Z)$

- The **global Markov assumption** for a Markov network *H* is

$$sep_H(X;Y|Z) \Rightarrow X \perp Y \mid Z$$

The BN Representation Theorem

If conditional independencies in BN are subset of conditional independencies in *P*	**Obtain** ⇒	Joint probability distribution: $P(X_1,\ldots,X_n) = \prod_{i=1}^{n} P\left(X_i \mid \mathbf{Pa}_{X_i}\right)$

Important because:
Independencies are sufficient to obtain BN structure *G*

If joint probability distribution: $P(X_1,\ldots,X_n) = \prod_{i=1}^{n} P\left(X_i \mid \mathbf{Pa}_{X_i}\right)$	**Obtain** ⇒	Then conditional independencies in BN are subset of conditional independencies in *P*

Important because:
Read independencies of *P* from BN structure *G*

Markov networks representation Theorem 1

If joint probability distribution *P*: $P(X_1,\ldots,X_n) = \frac{1}{Z}\prod_{i=1}^{m} \phi_i(\mathbf{D}_i)$	**Then** ⇒	*H* is an I-map for *P*

- If you can write distribution as a normalized product of factors $\Rightarrow$ Can read independencies from graph

What about the other direction for Markov networks ?

If H is an I-map for P **Then** joint probability distribution P:

$$P(X_1,\ldots,X_n) = \frac{1}{Z}\prod_{i=1}^{m}\phi_i(\mathbf{D}_i)$$

- Counter-example: $X_1,\ldots,X_4$ are binary, and only eight assignments have positive probability: (0,0,0,0) (1,0,0,0) (1,1,0,0) (1,1,1,0) (0,0,0,1) (0,0,1,1) (0,1,1,1) (1,1,1,1)
- For example, $X_1 \perp X_3 | X_2, X_4$:
  - E.g., $P(X_1=0|X_2=0, X_4=0)$
- But distribution doesn't factorize!!!

Markov networks representation Theorem 2 (Hammersley-Clifford Theorem)

If H is an I-map for P and P is a positive distribution **Then** joint probability distribution P:

$$P(X_1,\ldots,X_n) = \frac{1}{Z}\prod_{i=1}^{m}\phi_i(\mathbf{D}_i)$$

- Positive distribution and independencies $\Rightarrow$ P factorizes over graph

Representation Theorem for Markov Networks

If joint probability distribution P:
$$P(X_1,\ldots,X_n) = \frac{1}{Z}\prod_{i=1}^{m}\phi_i(\mathbf{D}_i)$$
Then → H is an I-map for P

If H is an I-map for P and P is a positive distribution **Then** → joint probability distribution P:
$$P(X_1,\ldots,X_n) = \frac{1}{Z}\prod_{i=1}^{m}\phi_i(\mathbf{D}_i)$$

Completeness of separation in Markov networks

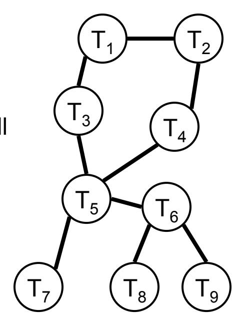
- **Theorem: Completeness of separation**
  - For "almost all" distributions that P factorize over Markov network H, we have that I(H) = I(P)
  - *"almost all" distributions*: except for a set of measure zero of parameterizations of the Potentials (assuming no finite set of parameterizations has positive measure)
- **Analogous to BNs**

What are the “local” independence assumptions for a Markov network?

- In a BN *G*:
  - local Markov assumption: variable independent of non-descendants given parents
  - d-separation defines global independence
  - Soundness: For all distributions:
- In a Markov net *H*:
  - **Separation** defines global independencies
  - What are the notions of local independencies?

Local independence assumptions for a Markov network

- **Separation** defines global independencies
- **Pairwise Markov Independence**:
  - Pairs of non-adjacent variables A,B are independent given all others
- **Markov Blanket**:
  - Variable A independent of rest given its neighbors



Equivalence of independencies in Markov networks

- **Soundness Theorem**: For all positive distributions *P*, the following three statements are equivalent:
  - *P* entails the global Markov assumptions
  - *P* entails the pairwise Markov assumptions
  - *P* entails the local Markov assumptions (Markov blanket)

Minimal I-maps and Markov Networks

- A fully connected graph is an I-map
- Remember minimal I-maps?
  - A "simplest" I-map $\rightarrow$ Deleting an edge makes it no longer an I-map
- In a BN, there is no unique minimal I-map
- Theorem: For positive distributions & **Markov network, minimal I-map is unique!!**
- Many ways to find minimal I-map, e.g.,
  - Take pairwise Markov assumption:
  - If *P* doesn't entail it, add edge:

How about a perfect map?

- Remember perfect maps?
  - independencies in the graph are exactly the same as those in *P*
- For BNs, doesn't always exist
  - counter example: Swinging Couples
- How about for Markov networks?

Unifying properties of BNs and MNs

- BNs:
  - give you: V-structures, CPTs are conditional probabilities, can directly compute probability of full instantiation
  - but: require acyclicity, and thus no perfect map for swinging couples
- MNs:
  - give you: cycles, and perfect maps for swinging couples
  - but: don't have V-structures, cannot interpret potentials as probabilities, requires partition function
- Remember PDAGS???
  - skeleton + immoralities
  - provides a (somewhat) unified representation
  - see book for details

What you need to know so far about Markov networks

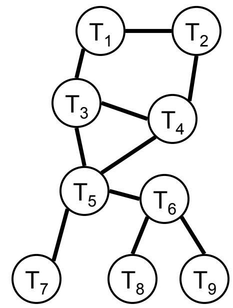
- Markov network representation:
  - undirected graph
  - potentials over cliques (or sub-cliques)
  - normalize to obtain probabilities
  - need partition function
- Representation Theorem for Markov networks
  - if P factorizes, then it's an I-map
  - if P is an I-map, only factorizes for positive distributions
- Independence in Markov nets:
  - active paths and separation
  - pairwise Markov and Markov blanket assumptions
  - equivalence for positive distributions
- Minimal I-maps in MNs are unique
- Perfect maps don't always exist

Some common Markov networks and generalizations

- Pairwise Markov networks
- A very simple application in computer vision
- Logarithmic representation
- Log-linear models
- Factor graphs

Pairwise Markov Networks

- All factors are over single variables or pairs of variables:
  - Node potentials
  - Edge potentials
- Factorization:



- Note that there may be bigger cliques in the graph, but only consider pairwise potentials

A very simple vision application

- Image segmentation: separate foreground from background
- Graph structure:
  - pairwise Markov net
  - grid with one node per pixel
- Node potential:
  - “background color” v. “foreground color”
- Edge potential:
  - neighbors like to be of the same class

Logarithmic representation

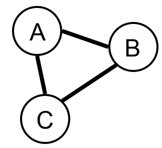
- Standard model: $P(X_1, \ldots, X_n) = \frac{1}{Z} \prod_{i=1}^{m} \phi_i(\mathbf{D}_i)$
- Log representation of potential (assuming positive potential):
  - also called the energy function
- Log representation of Markov net:

Log-linear Markov network (most common representation)

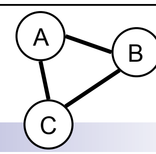
- **Feature** is some function f [**D**] for some subset of variables **D**
  - e.g., indicator function
- **Log-linear model** over a Markov network *H*:
  - a set of features $f_1[\mathbf{D}_1], \ldots, f_k[\mathbf{D}_k]$
    - each $\mathbf{D}_i$ is a subset of a clique in *H*
    - two f's can be over the same variables
  - a set of weights $w_1, \ldots, w_k$
    - usually learned from data
  - $P(X_1, \ldots, X_n) = \frac{1}{Z} \exp\left[\sum_{i=1}^{k} w_i f_i(\mathbf{D}_i)\right]$

Structure in cliques

- Possible potentials for this graph:



Factor graphs



- Very useful for approximate inference
  - Make factor dependency explicit
- Bipartite graph:
  - variable nodes (ovals) for $X_1,\ldots,X_n$
  - factor nodes (squares) for $\phi_1,\ldots,\phi_m$
  - edge $X_i - \phi_j$ if $X_i \in$ Scope$[\phi_j]$

Exact inference in MNs and Factor Graphs

- Variable elimination algorithm presented in terms of factors $\rightarrow$ exactly the same VE algorithm can be applied to MNs & Factor Graphs
- Junction tree algorithms also applied directly here:
  - triangulate MN graph as we did with moralized graph
  - each factor belongs to a clique
  - same message passing algorithms

Summary of types of Markov nets

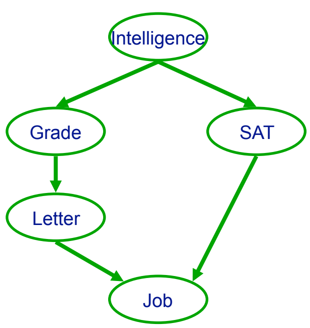
- Pairwise Markov networks
  - very common
  - potentials over nodes and edges
- Log-linear models
  - log representation of potentials
  - linear coefficients learned from data
  - most common for learning MNs
- Factor graphs
  - explicit representation of factors
    - you know exactly what factors you have
  - very useful for approximate inference

What you learned about so far

- Bayes nets
- Junction trees
- (General) Markov networks
- Pairwise Markov networks
- Factor graphs

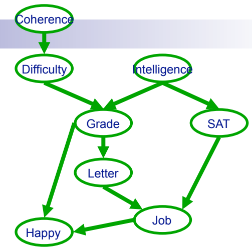
- How do we transform between them?
- More formally:
  - I give you an graph in one representation, find an **I-map** in the other

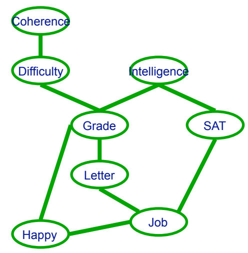
From Bayes nets to Markov nets



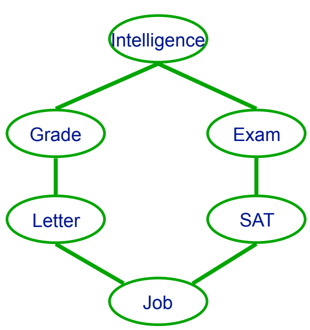
BNs → MNs: Moralization

- **Theorem**: Given a BN *G* the Markov net *H* formed by moralizing *G* is the *minimal I-map* for I(*G*)
- **Intuition**:
  - in a Markov net, each factor must correspond to a subset of a clique
  - the factors in BNs are the CPTs
  - CPTs are factors over a node and its parents
  - thus node and its parents must form a clique
- **Effect**:
  - **some** independencies that could be read from the BN graph become hidden



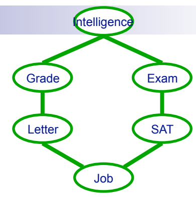


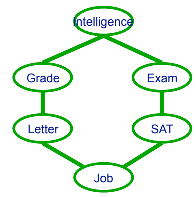
From Markov nets to Bayes nets



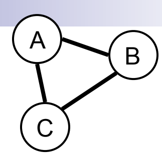
MNs → BNs: Triangulation

- **Theorem**: Given a MN *H,* let *G* be the Bayes net that is a *minimal I-map* for I(*H)* then *G* must be **chordal**
- **Intuition**:
  - v-structures in BN introduce immoralities
  - these immoralities were not present in a Markov net
  - the triangulation eliminates immoralities
- **Effect**:
  - **many** independencies that could be read from the MN graph become hidden





Markov nets v. Pairwise MNs



- Every Markov network can be transformed into a Pairwise Markov net
  - introduce extra "variable" for each factor over three or more variables
  - domain size of extra variable is exponential in number of vars in factor
- **Effect**:
  - any local structure in factor is lost
  - a chordal MN doesn't look chordal anymore

Overview of types of graphical models and transformations between them